Stability and Analysis of TCP Connections with RED Control and Exogenous Traffic

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Abstract : In this paper we study the stability and performance of a system involving several TCP connections passing through a tandem of RED controlled queues each of which has an incoming exogenous stream. The exogenous stream, representing the superposition of all incoming UDP connections into a queue, has been modeled as an MMPP stream. We consider both the TCP Tahoe and the TCP Reno versions. We start with the analysis of a single TCP connection sharing a RED controlled queue with an exogenous stream. The effect of the exogenous stream (which is almost always present in large networks) is to cause the system to converge to a stationary distribution from any initial conditions. This stability result holds good for any work conserving discipline. We also obtain the performance indices of the system; specifically the goodputs and the mean sojourn times of the various connections. The complexity involved in computation of performance indices for the system is reduced by approximating the evolution of the average queue length process of the RED queue by a deterministic ODE. Then, by using a decomposition approach of two time scales, we reduce the study of the system to that of a simplified one for which the performance measures can be obtained under stationarity. Finally, we extend the above results to the case when multiple TCP connections share a RED controlled queue with an exogenous stream and to the case when a TCP connection passes through several RED controlled tandem queues each of which has an incoming exogenous stream. We also consider an example of multiple TCPs passing through a tandem of queues. A number of simulation results have been provided which support the analysis.

Keywords: TCP protocol, two time scale decomposition, performance analysis, RED control.

1 Introduction

The TCP/IP (Transmission Control Protocol/ Internet Protocol) based Internet has emerged as the dominant networking technology today. Though there are congestion control mechanisms incorporated in TCP's sliding window protocol, yet it is beneficial to complement these with queue management mechanisms in routers. The recommended active queue management mechanism for Internet routers today is Random Early Detection (RED) [9], [17]. Many router manufacturers have already incorporated it in their products. The RED algorithm helps alleviate problems of global synchronization of TCP connections and reduces the bias against bursty connections. For bulk data transfer in the Internet, TCP is the most used protocol. Therefore to analyse the end-to-end performance of a TCP connection engaged in bulk data transfer, it is necessary to study a system with TCP connections and RED controlled routers.

In this paper we obtain the stability and the performance of a system involving TCP connections sharing RED controlled queues with exogenous streams. The basic model studied involves the output queue of a bottleneck router on the path of a TCP connection. This bottleneck queue is also shared by an exogenous stream, which represents the superposition of all UDP traffic streams coming into the queue. This reflects a typical scenario in tomorrow's Internet where router queues are expected to be shared by data traffic as well as by multimedia traffic. The exogenous stream has been modeled as an MMPP stream. The packet lengths of both the streams are assumed i.i.d with general distributions. Later on we also study the systems with multiple TCP connections passing through one router or a tandem of routers, each with RED control and exogenous streams.

For the different systems described above, we first prove the stability of the system and show the existence of a unique stationary distribution. We also obtain rates of convergence to the stationary distribution. We observe that the presence of exogenous traffic significantly alters the behavior of the system. Stochastic assumptions on the nature of the exogenous stream traffic and the random drop of packets of incoming streams by RED destroy the periodic evolution of TCP window size as observed by earlier studies [27]. This stability result is not available so far in the literature. The proofs for our stability results are valid for any work conserving discipline.

Next we obtain the performance indices for the system; specifically the mean sojourn times and the goodputs of the various streams. Computing these quantities even for the case when the incoming traffic has exponential interarrival times and packet lengths, is extremely intensive. Therefore, it is necessary to evolve a simple procedure. Following [38], we observe that the average queue length $\widehat{q}_n$ computed by the RED algorithm evolves much more slowly than the actual queue length $\{q_n\}$ and the counter $\{c_n\}$ processes. Then, using the decomposition approach of two time scales as in [38], we approximate the evolution of $\widehat{q}_n$ by the solution of an ODE (Ordinary Differential Equation). The decomposition approach then facilitates the reduction of the study of the system to a simplified one for which the performance indices can be obtained under stationary conditions. These can then be used to obtain the performance of the original system under stationary and transient conditions. The analysis has been carried out for all the different systems considered. We have validated all our approximations via simulations. Our analysis methods can be used for some other active congestion management algorithms where the system has two time scales.

In most of the theoretical studies on the performance of TCP congestion control mechanisms, the focus has primarily been on predicting the throughput of TCP connections in various scenarios like lossy links ([1], [25], [31]), long haul networks ([27]) and local area networks ([25]). In all these works one or more TCP connections pass through a single bottleneck link. There is no exogenous (UDP) stream in these studies. Also stability has not been proved. Altman *et al.* [2], Baccelli and Bonald [5], Bonald [7], Baccelli and Hong [6] are among the few papers that have actually considered exogenous flows. Stability results are available in [5] and [6]. However, we show stability when the window size is adaptively changing according to the Tahoe or the Reno algorithm, and we specifically consider the effect of the exogenous stream. We also prove stability for the system with RED control which has not been considered in any of these studies.

Considerable research effort has been directed recently towards analysing systems with TCP connections passing through routers employing active queue management (AQM). Some papers (for e.g., [13], [22] and [30]) have proposed modifications to extant AQM mechanisms like RED whereas others (for e.g, [14], [21], [22], [23], [28] and [29]), focus on an analysis of RED and its interaction with TCP. Misra *et al.*[29] is closest to our work. They consider multiple TCP connections passing through tandem routers. They use fluid models for the TCP connections and assume that loss indications from the network arrive back to the source as a Poisson process. We do not make any special assumptions on the loss behavior, except that the losses are assumed to be only due to the RED drop mechanism. Also we consider actual packet-level behavior instead of fluid models. The papers by Hollot *et al.* [21], [22] employ control theoretic principles in order to analyse general AQM schemes. Both the papers use a fluid model. In [23], ODE approximation is used as we do in this paper. However, they form a fluid model and our model is much more realistic. Also, they approximate the TCP rate process with an ODE while we will approximate the average queue length process of the RED algorithm by an ODE. Further more the stability issue has not been addressed in [23].

A preliminary version of this paper, without proofs was presented in [37]. Since then, we have extended this approach in various directions. [19] analyses the system when weighted round robin (WRR) scheduling policy is used on the TCP and UDP connections. [19] and [35] also study the system where the UDP streams have priority (preemptive and non-preemptive) over the TCP connections. In the present paper TCP connections are persistent. But, [19] and [20], extend this approach to the system where some of the TCP connections may be short lived and they arrive at the system as a Poisson process. [33] describes an approach to use this analysis to provide the desired quality of service to different TCP and UDP connections.

The paper is organised as follows. In section 2 we study a single queue with one or more TCP connections. We describe the model and provide the stability and the performance parameters. In section 3 the system involving a tandem of routers is analyzed and all the stability and performance results are obtained. Section 4 contains simulation results and their discussion. The few appendices attached provide proofs for the ODE approximations and also study those ODEs.

2 The single queue case

This section studies a single RED controlled queue shared by one or more TCP sessions and an exogenous stream. Section 2.1 introduces the model, section 2.2 proves the stability of the system with and without RED control. Section 2.3 carries out the performance analysis. Section 2.4 analyses the system with multiple TCP connections.

2.1 The Model

The basic single queue model (see Fig. 1) studied here involves a finite buffer output queue of a bottleneck router in the path of a TCP connection. An exogenous stream shares the queue with the TCP connection. The exogenous stream, which represents the superposition of all UDP streams coming into the queue, is modeled as an MMPP source. The MMPP model enables us to allow for the variability in the number of connections as also their burstiness. This is a commonly made assumption. The TCP source is assumed to transmit a long file (e.g., a long FTP session). Both the Reno and the Tahoe versions of the TCP protocol are considered. The packets of the TCP session arrive at the queue with a time delay of Δ_1 secs which includes the propagation and processing delays along the path as also the queueing delays in the intermediate nodes (later on we will explicitly study multiple queues). Thus it has a deterministic component Δ_1' and a random queueing delay component Δ_1''. Because of the queueing at intermediate nodes, the following realistic assumption will be made

$$\begin{aligned}&P(\text{interarrival time between two consecutive TCP packets}\\&\qquad\qquad\text{arriving at the bottleneck queue} > \delta) > p_T \end{aligned}\tag{1}$$

where δ and p_T are two (small) positive constants. Similarly the acknowledgments (ACKs) from the destination to the source take Δ_2 secs time, with $\Delta_2 = \Delta_2' + \Delta_2''$, Δ_2' being the deterministic component and Δ_2'' the random component. We assume Δ_1'' and Δ_2'' independent of each other and also of all other variables considered here. For the finite buffer queues considered here, Δ_1'' and Δ_2'' are both bounded and so are Δ_1 and Δ_2. For stability arguments we shall assume that the server at the queue serves at a unit rate. The service times (packet lengths) of the TCP and the MMPP streams will be assumed random, i.i.d. with general distributions. Let s_T and s be generic packet lengths of the TCP and the MMPP streams. The queue can store upto B bits of data. If a packet arrives and finds that the queue does not have enough space to store it completely, it will be lost. All the packets are served in an FCFS (first come first serve) fashion. Actually for stability we need only a work conserving discipline.

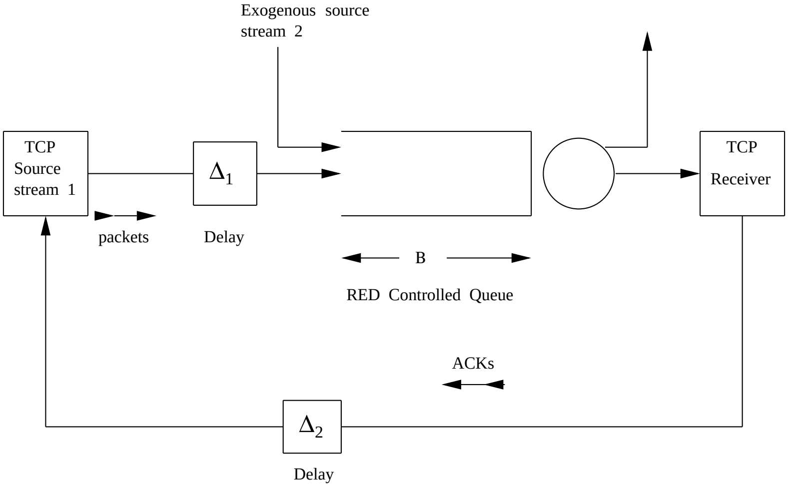

Figure 1: The basic model

For the basic model considered above, let at any time t, w_t be the window length of the TCP source, v_t the workload in the queue, h_t the slow start window threshold of the TCP source and q_t^T the number of TCP packets in the queue. Also let t_k be the time the TCP window is updated for the kth time. We denote by (w_k, v_k, h_k, q_k^T) the process (w_t, v_t, h_t, q_t^T) at times t_k^+. Let W_{max} be the maximum window size dictated by the receiver. Then $w_k \leq W_{max}$, $v_k \leq B$, $h_k \leq W_{max}$ and $q_k^T \leq W_{max}$. We will show in the next section that the processes $\{(w_k, v_k, h_k, q_k^T)\}$ and $\{(w_t, v_t, h_t, q_t^T)\}$ have unique stationary distributions.

We briefly describe here the window control mechanisms of TCP Tahoe and TCP Reno. For details, see [40].

For TCP Tahoe when new data is acknowledged (new ACKs are received) the sender recomputes its window size as:

$w_{k+1} = w_k + 1$ if $w_k < h_k$ (slow start phase)

$w_{k+1} = w_k + 1/w_k$ if $w_k \geq h_k$ (congestion avoidance phase).

If the sender receives a duplicate ACK, then the window size is not increased.

When a packet loss is detected (either by the reception of 3 duplicate ACKs or by a timeout) the sender immediately retransmits the lost packet and sets

$w_{k+1} = 1, \qquad h_{k+1} = max(w_k/2, 2)$.

TCP Reno is an improved version of TCP Tahoe and follows the window adaptation algorithm described below.

When a new ACK arrives the window update is done as in Tahoe. When a packet loss occurs Reno reacts to it depending upon whether the loss indication is due to three duplicate ACKs received or due to a timeout. When 3 duplicate ACKs are received Reno sets in motion the fast retransmit and fast recovery mechanisms:

(1) When the third duplicate ACK arrives, set

$h_{k+1} = max(w_k/2, 2), \qquad w_{k+1} = h_{k+1} + 3$

and retransmit the missing packet.

(2) Each time an additional duplicate ACK arrives (requesting the same packet)

$w_{k+1} = w_k + 1.$

and transmit a segment if allowed by the window.

If three duplicate ACKs do not arrive back at the source by the time the timer times out

$w_{k+1} = 1, \qquad h_{k+1} = 2$

and reinitiate slow start.

The retransmission timeout (RTO) is calculated dynamically based on measurements of both the mean and the variance of the RTT (round trip time) measurements (see [40] for various schemes). Every time a packet is sent a timer waits for the amount of time equal to the most recently updated RTO value for the ACK to come back, before deciding to retransmit the packet.

2.1.1 The RED Algorithm

Consider the queue with a finite buffer of size B as in Fig.1. In this section we employ the RED algorithm to control the queue. The RED algorithm is explained below.

Let q_n be the total queue length at the time of the nth packet arrival to the queue. Let $\widehat{q}_n$ be the average queue length (to be defined below) at the nth packet arrival. The nth packet will be discarded with probability p_n, which depends upon $\widehat{q}_n$ and the past history. By defining p_n appropriately, the RED algorithm controls congestion in the router before its queue overflows. Fix the constants p_{max}, T_{max} and T_{min} appropriately such that $0 \leq T_{min} \leq T_{max} \leq B$. Then p_n is defined as

If $\widehat{q}_n < T_{min}, \qquad p_n = 0,\ c_n = -1.$

If $\widehat{q}_n > T_{max}, \qquad p_n = 1,\ c_n = 0.$

If $T_{min} \leq \widehat{q}_n \leq T_{max},$

$c_n = c_{n-1} + 1,$
$C_n = (T_{max} - T_{min})/[\ p_{max}(\widehat{q}_n - T_{min})\ ],$
$p_n = 1/(C_n - c_n).$

With probability p_n, discard the packet and set $c_n = 0$. With probability $1-p_n$, queue the packet.

The average queue length $\widehat{q}_n$ is defined as
$\widehat{q}_n = (1-\beta)\widehat{q}_{n-1} + \beta q_n$, if $q_n > 0$
$\widehat{q}_n = (1-\beta)^m \widehat{q}_{n-1}$, otherwise
where $m =$ (idle time during the interarrival time of the nth packet)$/\bar{s}$
$\bar{s} =$ typical transmission time for a small packet.

and $0 < \beta < 1$ is an appropriately fixed constant. Typically β is chosen relatively small, e.g. Floyd and Jacobson[17] propose using $\beta \geq 0.001$ and use $\beta = 0.002$ in their simulations.

In the above algorithm it is possible that p_n can become negative or greater than 1. In that case we set $p_n = 0$ or 1 as the case may be. Also, it is possible that the algorithm decides that an arriving packet should be admitted but the workload exceeds B. In that case the packet should be discarded and we set $c_n = 0$.

2.2 Stability

In this section we obtain stability results for the models described in the previous section. In section 2.2.1 we provide stability proofs for the single TCP, single queue case when the queue is not RED controlled. In section 2.2.2 we provide stability proofs for the system with RED control. Our proofs hold for any work conserving queueing discipline. In particular the system with time priority to TCP or UDP is allowed. We can also give higher space priority to the UDP stream (but not to the TCP in following proofs).

2.2.1 Stability without RED control

Consider the case when the buffer size B (Figure 1) is infinite. Because there are no packet drops, the window size after some initial change stays constant at W_{max}. The stability of such a queueing system with fixed window size source is shown in [8], [36] and appendix A.

In both the theorems below we prove stability when the exogenous stream is Poisson with rate λ and the buffer is finite (can hold B bits at a time). Immediately thereafter we extend the arguments to the MMPP case. RTO is a random variable which denotes the generic retransmission timeout of a TCP source packet. It is changing with time and we assume that it is upper bounded by some fixed $\gamma < \infty$. Because the exogenous stream represents the superposition of all other UDP streams coming into the queue and because each UDP stream can have varying packet sizes, the distribution function of its packet size can be assumed to be absolutely continuous with respect to Lebesgue measure in the interval $[0, \epsilon]$ for some fixed $\epsilon > 0$. (It will be especially true for high speed networks). In the following we will make a some what weaker assumption.

Theorem 1 proves the stability for the Tahoe version while Theorem 2 proves the stability for the Reno version of TCP. The details of these protocols are available in [35],

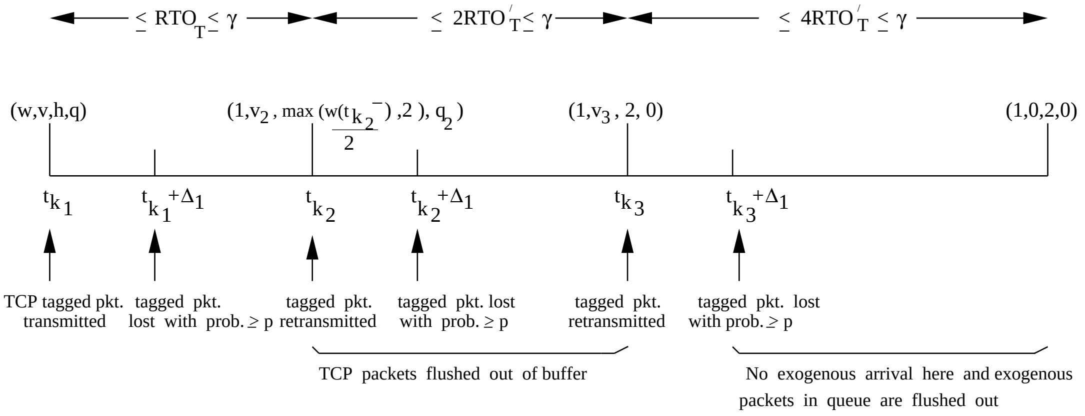

Figure 2: Diagram illustrating the sequence of states in stability proof: of Theorem 1

some of which are used in the proof (e.g., exponential back off). We also obtain the rates of convergence to the stationary distributions.

Theorem 1 (Stability for the TCP Tahoe case) *Assume*

(1) *the distribution function of s has an absolutely continuous component with respect to the Lebesque measure in the interval $[0, \epsilon]$ for some fixed $\epsilon > 0$,*

(2) *the buffer size $B < 2\Delta'$ where $\Delta' = \Delta_1' + \Delta_2'$,*

(3) $P[s_T \geq \alpha_{min}] > 0$ *for some* $\alpha_{min} > 0$.

Then the processes $\{(w_k, v_k, h_k, q_k^T)\}$ and $\{(w_t, v_t, h_t, q_t^T)\}$ have unique stationary distributions. Also, starting from any arbitrary initial state, these processes converge to their stationary distributions in total variation.

Proof. Whenever $(w_k, v_k, h_k, q_k^T) = (1, 0, 2, 0)$ and a TCP packet is transmitted from the source and no TCP packets or ACKs are propagating, the system can be considered to regenerate (in fact we could require the queue to be empty only when the transmitted TCP packet has propagated for time $\Delta' - \delta_1$ where δ_1 is a fixed small constant defined below). Thus if we show that the intervisit time τ to this state satisfies $E[\tau] < \infty$, by regenerative process theory the process $\{(w_k, v_k, h_k, q_k^T)\}$ has a unique stationary distribution π. Also, if we show that τ has an aperiodic distribution, then this process converges to π in total variation starting from any arbitrary initial state (Asmussen [4], Chapter V).

We start with showing $E[\tau^\beta] < \infty$ for any $\beta > 0$. By the geometric coin tossing argument, it is sufficient to show that starting in any arbitrary initial state (w, v, h, q) in a finite (upper bounded) time the state $(1, 0, 2, 0)$ can be reached with a positive probability (lower bounded away from zero).

Let, at time t_{k_1}, the system be in state (w, v, h, q) $(w \geq 1,\ v \geq 0,\ h \geq 2,\ q \geq 0)$ and the source transmits a packet called the *tagged* packet (in any case within a finite time $\leq \gamma$ a packet will leave the source). Let $\delta_1 = \delta/3$ (where δ is defined in (1)).

In the following, Δ_1 will be any fixed realization of the forward propagation delay. At time $t_{k_1} + \Delta_1 - \delta_1$ let x be the workload in the queue. With probability $\geq p_T$ no other TCP packet arrives at the queue during time $[t_{k_1} + \Delta_1 - \delta_1, t_{k_1} + \Delta_1]$. If $x - \delta_1 > B - \alpha_{min}$ (for the case when $\alpha_{min} > \delta_1$) then with probability $\geq P[s_T \geq \alpha_{min}].p_T > 0$ the tagged packet is dropped on arrival at the queue. Now assume $x \leq B - (\alpha_{min} - \delta_1)^+$ (for both cases when $\alpha_{min} > \delta_1$ and $\alpha_{min} \leq \delta_1$). Choose an integer N_0 such that $\frac{B+\delta_1}{N_0} < \epsilon$. Then by assumptions (1) and (3) the tagged packet gets discarded on arrival at the queue with probability lower bounded by

$$p \triangleq \{ \inf_{0 \leq x \leq B-(\alpha_{min}-\delta_1)^+} P[\frac{B - \alpha_{min} - (x-\delta_1)^+}{N_0} < s \leq \frac{B-(x-\delta_1)^+}{N_0}]\}^{N_0} \; . \; \frac{e^{-\lambda\delta_1}(\lambda\delta_1)^{N_0}}{N_0!} \; .$$

$$P[s_T \geq \alpha_{min}] \; .p_T \; > \; 0.$$

The second term on the right in the above expression is the probability that $N_0 > 0$ exogenous packets arrive to the queue during time $[t_{k_1} + \Delta_1 - \delta_1, t_{k_1} + \Delta_1]$. The first term is a lower bound on the probability that they together occupy the space in the buffer such that at time $t_{k_1} + \Delta_1$ the space left is less than α_{min}. The term inside the inf is strictly positive because of (1) and because we have chosen N_0 large enough that $\frac{B+\delta_1}{N_0} < \epsilon$, and the inf is also strictly positive because by assumption (1) we are taking infimum of a strictly positive continuous function on a compact set.

Thus within a finite time (upper bounded by the retransmission timeout of the tagged packet denoted by $RTO_T \leq \gamma$), say at time t_{k_2}, 3 duplicate ACKs arrive at the source or a timeout occurs (whichever is smaller) and the system state becomes $(1, v_2, max(\frac{w_{t_{k_2}^-}}{2}, 2)$, $q_2)$ for some $v_2 \geq 0$ and $q_2 \geq 0$. It is possible that $w_{t_{k_2}^-} > w$ because of successful transmissions before the tagged packet during time $[t_{k_1}, t_{k_2}]$. In the mean time, because of successful packet transmissions before the tagged packet, RTO could have been updated to $RTO_T{}'$. Due to exponential backoff, the RTO for the tagged packet now becomes $2RTO_T{}'$. The tagged packet is immediately retransmitted. The retransmitted tagged packet on arrival at the queue after time Δ_1, will again be lost with a probability $\geq p$. Thus, after time $2RTO_T{}' \leq \gamma$ (timeout occurs), the system goes to state $(1, v_3, 2, 0)$ (at time say t_{k_3}), for some $v_3 \geq 0$, with probability $\geq p$. TCP packets get flushed out of the buffer because of assumption (2) and because $RTO_T{}' \geq \Delta'$. The tagged packet (RTO is now $4RTO_T{}'$ (exponential backoff)) is again retransmitted and with a probability $\geq p$ is lost. Also with probability $e^{-\lambda(4RTO_T{}' - \Delta_1)} \geq e^{-\lambda(\gamma - \Delta_1')}$ no exogenous stream arrival occurs in time $4RTO_T{}' - \Delta_1$. Thus by assumption (2), after time $4RTO_T{}' \leq \gamma$ we reach the state $(1, 0, 2, 0)$ from $(1, v_3, 2, 0)$ with probability $\geq e^{-\lambda(\gamma-\Delta_1')}p$ (exogenous packets are flushed out of the buffer because of assumption (2) and because $RTO_T{}' \geq \Delta'$). Thus we have shown that starting from any initial state (w, v, h, q) in a finite upper bounded time ($\leq 3W_{max}$) we can obtain a regeneration with a positive (bounded away from zero) probability. Therefore, by a geometric coin tossing argument, we obtain $E[e^{\theta\tau}] < \infty$ for some $\theta > 0$.

Next we show the aperiodicity of the distribution of τ. Specifically, we show that with positive probability we can reach $(1, 0, 2, 0)$ starting from $(1, 0, 2, 0)$ within any number of steps ≥ 3. We have shown above that, starting from $(1, 0, 2, 0)$ we can reach the same state in three steps with a positive probability. Arguments for the other cases (say for four steps) are exactly similar.

Next let us consider the continuous time process $\{(w_t, v_t, h_t, q_t^T)\}$. Since the delays Δ_1, Δ_2 and the RTT, RTO involved in the system are upper bounded, and the queueing delay in the queue is $\leq B$, $t_{k+1} - t_k$ is upper bounded as well. Therefore, taking the renewal epochs of $\{(w_k, v_k, h_k, q_k^T)\}$ as the renewal epochs of $\{(w_t, v_t, h_t, q_t^T)\}$ also, the regeneration

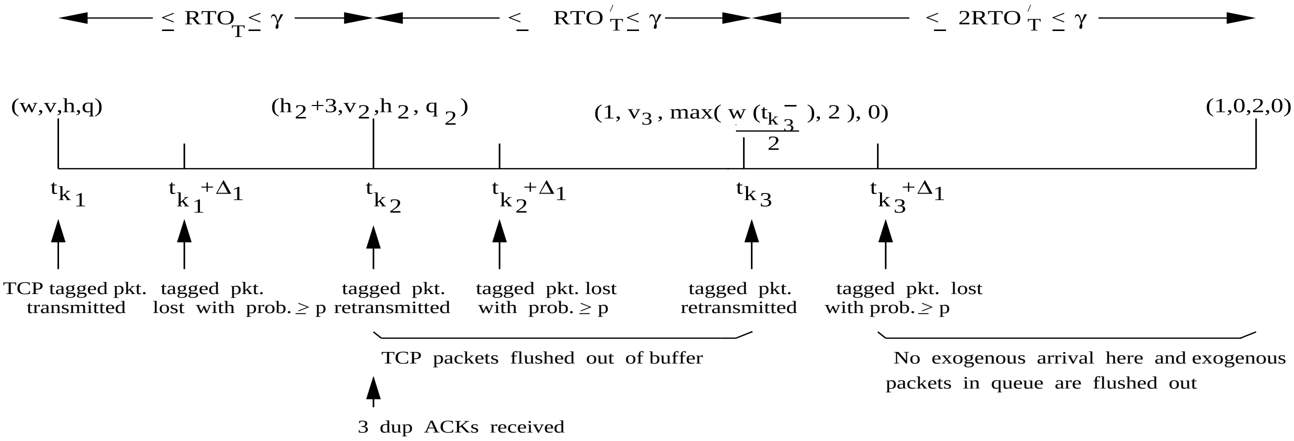

Figure 3: Diagram illustrating the sequence of states in stability proof of Theorem 2

length $\hat{\tau}$ for this process also satisfies $E[e^{\theta' \tau}] < \infty$ for some $\theta' > 0$. Also, we have seen above that a regeneration length involves several interarrival times of the exogenous stream (which has exponential distribution). Thus $\hat{\tau}$ has a spreadout distribution. Hence from Asmussen[4], p.126, we obtain a unique stationary distribution $\hat{\pi}$. The convergence to $\hat{\pi}$, starting from any initial conditions, also holds. □

Next consider the generalization when the stream 2 is an MMPP with the modulating chain $\{y_t\}$ a finite state irreducible Markov chain. This implies that the probability $P_t(i, i_0)$ of the event that y_t reaches from any state i to any other state i_0 in time t is positive for all i and $t > 0$. When $y_t = i$, the instantaneous arrival rate for stream 2 is denoted by λ_i. Assume that $\lambda_i > 0$ at least for some state i. Fix a state i_0 of y_t. Let $y_k = y_{t_k}$. Then the process $\{(w_k, v_k, h_k, q_k^T, y_k)\}$ has a regeneration whenever it reaches $(1, 0, 2, 0, i_0)$. Consider the epoch t_k of $\{(w_k, v_k, h_k, q_k^T)\}$ when the process reaches $(1, 0, 2, 0)$. Then as mentioned above irrespective of the state y_{t_k} we can ensure $y_{t_k+\delta} = i_0$ with a positive probability for an arbitrarily small fixed constant $\delta > 0$ and that no exogenous arrivals occur during time $[t_k, t_k + \delta]$. Such an epoch $t_k + \delta$ can be taken as the regeneration epoch for the process $(w_k, v_k, h_k, q_k^T, y_k)$. This provides us with the finiteness of the βth moment of the regeneration length of the process $(w_k,\ v_k,\ h_k,\ q_k^T,\ y_k)$ for any $\beta > 0$. Similarly we can obtain the result for the continuous time process.

From Kalashnikov[24], $E[e^{\theta \tau}] < \infty$ for some $\theta > 0$ provides us with the rate of convergence

$$\sup_A \Big| P[(w_k, v_k, h_k, q_k^T) \in A] - \pi(A) \Big| < c.e^{-k\bar{\theta}} \tag{2}$$

where π is the unique stationary distribution of the process and c and $\bar{\theta}$ are some positive constants. Also since all the components of the process are bounded, we obtain a Functional Central Limit Theorem (FCLT) and other strong limit theorems (see Sharma[34]). Similarly, we also obtain the rate of convergence (2) and the FCLT for the continuous time process.

Theorem 2 (Stability for the TCP Reno case) *Under assumptions* (1) *and* (3) *of Theorem* 1 *and the assumption*
$(2')$ $B < \Delta'$ *where* $\Delta' = \Delta'_1 + \Delta'_2$,

the processes $\{(w_k, v_k, h_k, q_k^T)\}$ and $\{(w(t), v_t,\ h_t, q_t^T)\}$ have unique stationary distributions. Also, starting from any arbitrary initial state, these processes converge to their stationary distributions in total variation.

Proof. As for the TCP Tahoe case we show here that starting in any arbitrary initial state (w, v, h, q) in a finite upper bounded time the state $(1, 0, 2, 0)$ can be reached with a positive probability.

As in Theorem 1 we consider an epoch in the system evolution, t_{k_1}, when the system is in state (w, v, h, q). The case when $w = 1$ can be easily argued following the lines of reasoning in Tahoe (with a window size of 1). We now consider the case when $w > 1$. The TCP source transmits a tagged packet which arrives at the queue Δ_1 secs later. With a probability $\ge p$ (just as in arguments for the TCP Tahoe case) this packet on arrival sees the buffer full and is lost. Within a finite time (upper bounded by $RTO_T \le \gamma$) we either get 3 duplicate ACKs for the tagged packet or a timeout occurs. If a timeout occurs, after time RTO_T, the system reaches the state $(1, v_2, max(w_{t_{k_2}^-}/2, 2), q_2)$ where $v_2 \ge 0$ and $q_2 \ge 0$. Then by arguing as we have been doing for Tahoe (with a window size of 1) we can ensure that we reach the state $(1, 0, 2, 0)$ within a finite upper bounded time with a positive probability.

If 3 duplicate ACKs are received at the source, at time say t_{k_2}, we reach the state $(h_2 + 3, v_2, h_2, q_2)$ where $v_2 \ge 0$, $q_2 \ge 0$ and $h_2 = max(w_{t_{k_2}^-}/2, 2)$. It is possible that $w_{t_{k_2}^-} > w$ because of successful transmissions before the tagged packet during time $[t_{k_1}, t_{k_2}]$. The TCP source retransmits the tagged packet (the RTO of the tagged packet is now $RTO_T{}'$) which sees a full buffer with a probability $\ge p$ and again gets lost. We describe now the general sequence of events that take place in the RTO period ($RTO_T{}'$) of the tagged packet according to the Reno algorithm. With additional duplicate ACKs for the same segment (owing to packets and ACKs in the round trip pipe and in the queue), the congestion window increases by 1 per additional duplicate ACK received for $t > t_{k_2}$. When the congestion window regains its original value, $w_{t_{k_2}^-}$, and increases beyond, newer packets are transmitted by the source (the number of such packets is $\le w_{t_{k_2}^-} \le W_{max}$, being limited by the number of additional duplicate ACKs received) and with the same positive probability ($\ge p$ as above) each such packet is lost. At the queue, the interarrival time between these packets is at least δ with probability $\ge p_T$, and by choosing δ_1 as in the arguments for Tahoe, the stated lower bound probabilities can be obtained. Then once time out occurs, say at time t_{k_3}, we reach the state $(1, v_3, max(w_{t_{k_3}^-}/2, 2), 0)$ (where $v_3 \ge 0$) (TCP packets are flushed out by assumption $(2')$ and because $RTO_T{}' \ge \Delta'$) and slow start is initiated. Thus from state $(h_2 + 3, v_2, h_2, q_2)$ we reach state $(1, v_3, max(w_{t_{k_3}^-}/2, 2), 0)$ within time $RTO_T{}' \le \gamma$ with probability $\ge p^{w_{t_{k_2}^-}}.\ p \ge p^{W_{max}+1}$. At t_{k_3}, the TCP source retransmits the tagged packet (RTO is now $2RTO_T{}'$). With probability $\ge p$ this packet on arrival at the queue also gets lost. With a probability $e^{-\lambda(2RTO_T{}'-\Delta_1)} \ge e^{-\lambda(\gamma-\Delta_1')}$ no arrival occurs from the exogenous stream in time $2RTO_T{}' - \Delta_1$. Using assumption $(2')$ (exogenous stream packets are flushed out of the buffer), with probability $\ge e^{-\lambda(\gamma-\Delta_1')}p$ we reach the state $(1, 0, 2, 0)$ (starting from the state $(1, v_3, max(w_{t_{k_3}^-}/2, 2), 0)$ after the timer times out.

Aperiodicity arguments are exactly the same as for the TCP Tahoe case. The arguments for the continuous time process $\{(w_t, v_t, h_t, q_t^T)\}$ are also the same as for the Tahoe case. $\square$

The generalization to the case when the input traffic is MMPP is handled the same way as for Tahoe. The rates of convergence to the stationary distribution for the processes $\{(w_k, v_k, h_k, q_k^T)\}$ and $\{(w_t, v_t, h_t, q_t^T)\}$ hold as in Tahoe.

2.2.2 Stability with RED control

In this section we show the stability of the system in Fig.1 when the queue is RED controlled. We assume that there are no packets lost except when allowed by RED. We consider the process $\{(w_k, v_k, h_k, q_k^T, \widehat{q}_k, c_k)\}$ at times t_k (there is a slight abuse of notation as compared to section 2.2.1 where $\widehat{q}_n$ and c_n were at nth arrival epoch to the queue). We showed in section 2.2.1 that intervisit times to the epochs when $\{(w_k, v_k, h_k, q_k^T)\}$ visits state $(1, 0, 2, 0)$ and a TCP packet is being transmitted from the source and no TCP packets and ACKs are propagating, has finite moments. Now we consider the intervisit times to the epochs when $\{(w_k, v_k, h_k, q_k^T, \widehat{q}_k, c_k)\}$ visits state $(1, 0, 2, 0, 0, -1)$ and a TCP packet is being transmitted from the source and no TCP packets and ACKs are propagating.

Let the process be in state $(1, 0, 2, 0, \widehat{q}, c)$ at time $t_0 = 0$, where $\widehat{q}$, c are any permissible values. We show that state $(1, 0, 2, 0, 0, -1)$ will be visited in a finite (upper bounded) time with a positive probability (lower bounded away from 0). A TCP packet is transmitted by the source. If no stream 2 packet arrives, this packet will reach the queue at time Δ_1, will be transmitted (with probability $1 - p(\widehat{q})$) and its ACK reaches the source at time $\Delta_1 + \bar{\alpha} + \Delta_2$, where $\bar{\alpha}$ was its packet size. During this cycle time, the idle time in the queue was $\Delta_1 + \Delta_2$ and hence the average queue length becomes $(1-\beta)^{\Delta_1/\bar{s}}(1-\beta)^{\Delta_2/\bar{s}}\widehat{q} \le (1-\beta)^{\Delta_1'/\bar{s}}(1-\beta)^{\Delta_2'/\bar{s}}\widehat{q}$, where $\bar{s}$ is the transmission time of a small packet. This event happens with a positive probability. During the next cycle, two TCP packets are sent and in this cycle again $\Delta_1 + \Delta_2$ is the idle time (if no stream 2 packet arrives) and hence the average queue length decreases again. Thus, with a positive probability, for reasonably large values of Δ_1' and Δ_2' (and hence Δ_1 and Δ_2) compared with $\bar{s}$, in a few cycles, $\widehat{q}_k$ can be made smaller than $\epsilon > 0$ where ϵ is an appropriately small constant (since $h_k = 2$ and we will be in congestion avoidance mode this event becomes more likely). Then we declare that $\widehat{q}_k = 0$ (this will happen in a practical system due to finite precision arithmetic) and $c_k = -1$. But now w_k would have increased. Thus with a positive probability, within a few cycles we can reach the state $(w, 0, 2, 0, 0, -1)$ for some $w > 1$.

We now provide arguments when the source implements TCP Tahoe. When the next (tagged) TCP packet reaches the queue, with a positive probability the queue is full (with stream 2 packets) and that TCP packet is dropped. This makes $w_k = 1$, $v_k = v$, $q_k^T = q$ (for some $v \ge 0$, $q \ge 0$) once 3 duplicate ACKs come back or a timeout occurs (whichever occurs earlier). Thereafter, if the tagged packet is dropped on successive retransmissions (this happens with a positive probability), then reasoning as in Theorem 1 after one or two retransmissions the system state will be $(1, 0, 2, 0, 0, -1)$. Since β is small, this can be ensured with a small increase in $\widehat{q}_k$ which may become less than ϵ with some idle time in the cycle. Thus with a positive probability (lower bounded away from zero, independently of $\widehat{q}$, c) the system can reach $(1,0,2,0,0,-1)$ from any state $(1,0,2,0,\widehat{q},c)$ in a finite (upper bounded) time.

Now let's consider the case when the source implements TCP Reno. A (tagged) TCP packet is transmitted and once it reaches the queue it is dropped with a positive probability. Then once the next update occurs, $w_k = 1$ or $w_k = h+3$ depending upon whether timeout occurs or 3 duplicate ACKs arrive at the source ($h = max(w'/2, 2)$ where w' is the window size just before 3 duplicate ACKs are received). As in the TCP Tahoe case, $v_k = v$, $q_k^T = q$ (for some $v \geq 0$, $q \geq 0$). Then if the tagged packet is dropped on successive retransmissions (this happens with a positive probability), and reasoning as in Theorem 2 after a few retransmissions the system state will be $(1,0,2,0,0,-1)$. And as before if β is small, this can be ensured with a small increase in $\widehat{q}_k$ which may become less than ϵ with some idle time in the cycle. Thus with a positive probability (lower bounded away from zero, independently of $\widehat{q}$, c) the system can reach $(1,0,2,0,0,-1)$ from any state $(1,0,2,0,\widehat{q},c)$ in a finite upper bounded time. This provides the finiteness of moments of regeneration length for this process.

The exponential backoff strategy for the retransmission times doubles the RTO everytime packets are retransmitted after timeouts. This increases the idle times in one RTO and hence increases the rate of decrease of $\widehat{q}$ in one cycle in the above argument.

2.3 Performance analysis

Computing performance indices for the system in Fig. 1 is very expensive. From Sharma et al.[38] we know that the same is true for a RED controlled queue also. Therefore, computing the performance of a TCP connection with a RED control is rather more difficult. However Sharma et al.[38] have obtained approximations for a RED controlled queue which are far more amenable to computation. We use a similar approach here.

The basic idea in this approximation is to realise that the process $\{\widehat{q_n}\}$ evolves much more slowly (because of the small β used in practical systems) as compared to the other components $(w_k, v_k, h_k, q_k^T, c_k)$. When $\widehat{q}_n \equiv \widehat{q} \leq T_{min}$, then the fact that the system is stable (has a stationary distribution) is shown in [36] and [8] for propagation delay $\Delta = 0$ ms. When $\widehat{q}_n \equiv \widehat{q} > T_{min}$, then every packet (TCP or UDP) entering the queue gets dropped with probability $p(\widehat{q})$ (according to the RED algorithm). Then similar arguments as in Theorems 1 and 2 show that the system processes $\{(w_k, v_k, h_k, q_k^T)\}$ and $\{(w_t, v_t, h_t, q_t^T)\}$ have unique (quasi)stationary distributions. We do not need the assumptions (1) - (3) (or (1), (2′), (3)) of the Theorems 1 and 2 in order to show this (for both TCP Tahoe and Reno). This is because the events mentioned in the proofs of Theorems 1 and 2 for regenerations to occur, now occur because TCP packets are getting lost with probability $p(q)$. Therefore, if we can approximately study the evolution of $\{\widehat{q_n}\}$, then we can obtain the (quasi)stationary distributions of the other components at that time. We will observe that this decomposition of the system reduces the complexity of obtaining the performance measures substantially. In section 4 we will verify the accuracy of this approach via simulations.

In section 2.3.1 we study the ODE approximation of the process $\{\widehat{q}_n\}$ for the single queue. For solving the ODE we will need $E_\pi[q|\hat{q}]$, the quasistationary mean of q (total queue length) given $\widehat{q}_n \equiv \widehat{q}$. Therefore, we obtain approximations for $E_\pi[q|\hat{q}]$ also. In the process we will also obtain the other quasistationary quantities of interest. The results are extended to the case of multiple TCP sessions in section 2.3.2.

2.3.1 The ODE approximation

We study the evolution of the process $\{\widehat{q}_n\}$ by approximating it by an ODE. If we ignore the possibility of $q_n = 0$, the process evolves as

$$\widehat{q}_{n+1} = \widehat{q}_n + \beta(q_{n+1} - \widehat{q}_n). \tag{3}$$

Since the RED control will be effective only when the incoming traffic intensity is high, ignoring $q_n = 0$ will not have much impact on the system dynamics under this condition. Taking expectations under stationarity in (3) we obtain $E_\pi[\widehat{q}^\beta] = E_\pi[q]$ where π represents the stationary distribution at an arrival instant.

Define the process $\{\widehat{z}_t^\beta, t \geq 0\}$ as

$$\widehat{z}_t^\beta = \widehat{q}_{\lfloor \frac{t}{\beta} \rfloor}, \qquad \widehat{z}_0^\beta = \widehat{q}_o. \tag{4}$$

We show in appendix A (under some conditions) that as $\beta \downarrow 0$ for any $0 < T < \infty$,

$$\sup_{0 \leq t \leq T} \left| \widehat{z}_t^\beta - z_t \right| \xrightarrow{P} 0 \tag{5}$$

($\xrightarrow{P}$ denotes convergence in probability) where $\{z_t, t \geq 0\}$ is the solution of the ODE

$$\frac{dz_t}{dt} = E_\pi[q|z_t] - z_t, \qquad z_0 = \widehat{q}_o = a \tag{6}$$

where $E_\pi[q|z_t]$ is the (quasi)stationary mean of the process $\{q_n\}$ when $\widehat{q}_n \equiv z_t$. Since in practical systems β used is small, $\{z_t\}$ thus obtained can provide a useful approximation.

By the boundedness of $\widehat{z}_t^\beta$, (5) implies the convergence of all moments of $\widehat{z}_t^\beta$ to that of z_t. In particular it implies that $\text{var}(\widehat{z}_t^\beta) \longrightarrow 0$ as $\beta \downarrow 0$ for any $t \geq 0$. The following useful results can also be obtained. Let τ_{min}^β and τ_{max}^β be the first time $\{\widehat{q}_n^\beta\}$ reaches T_{min} and T_{max}. These are important epochs in the sense that τ_{min}^β tells the time the queue is entering the mild congestion region. From (5) we also obtain $\tau_{min}^\beta \xrightarrow{P} \widehat{T}_{min}$ and $\tau_{max}^\beta \xrightarrow{P} \widehat{T}_{max}$ whenever T_{min} and $\widehat{T}_{max}$ are finite where $\widehat{T}_{min}$ and $\widehat{T}_{max}$ are the epochs when z_t reaches T_{min} and T_{max} respectively.

This motivates us to obtain the solution of the ODE (6). In the appendix we show that (6) has a unique, bounded, continuous solution, extendable to infinity for all initial conditions $0 \leq a \leq T_{max}$. Also, we show that (6) has a unique global attractor q^*, which can be easily obtained by either iteratively finding the zero of the right side of (6) or by numerically finding the solution of the ODE until it reaches the attractor (approximately). To obtain the solution of (6) for any initial a, we can use standard numerical techniques if we know $E_\pi[q|z_t]$ for all $0 \leq z_t \leq T_{max}$. Below we will approximately obtain $E_\pi[q|z_t]$. The accuracy of the solution so obtained will be verified via simulations in section 4. Furthermore, if $a < T_{min}$ then until z_t reaches T_{min}, $E_\pi[q|z_t]$ is independent of z_t ($=E_\pi[q]$ say) and the solution of the ODE simply becomes

$$z_t = ae^{-t} + E_\pi[q](1 - e^{-t}). \tag{7}$$

If $a < T_{min}$ and $E_\pi[q] < T_{min}$ then this solution is valid for all t.

Next we explain how to approximate the stationary distribution and moments of the system parameters via the solution of (6). Let $\widehat{q}^\beta$ denote $\{\widehat{q}_n^\beta\}$ under stationarity. In appendix B we show that $\widehat{q}^\beta$ converges in distribution to q^* as $\beta \downarrow 0$. Also, since $\widehat{q}^\beta$ is bounded, this implies $E[(\widehat{q}^\beta)^\alpha] \longrightarrow (q^*)^\alpha$ for any $\alpha > 0$. Thus for small β, $E[q^\beta]$ can be approximated by q^* which is much easier to compute. Similarly, one can approximate the stationary distribution of $(w_n^\beta, v_n^T, h_n^\beta, q_n^{T,\beta}, c_n^\beta)$ with the stationary distribution conditioned on $\widehat{q}^\beta = q^*$. From this one can also compute the stationary distribution of packet loss.

2.3.2 Quasistationary moments

As mentioned earlier, the process $\{\widehat{q}_n\}$ changes very slowly as compared to the other components of the process. Thus if $\widehat{q}_n = \widehat{q}$ then the other faster moving components attain a (quasi) stationarity dependent upon $\widehat{q}$. In this section we obtain approximations for the moments of these quasistationary quantities. In the process we will also obtain $E_\pi[q|\widehat{q}]$ for different values of $\widehat{q}$ which will be used in the ODE (6).

In what follows we assume that any packet loss is due to RED control and not due to buffer overflow. If T_{max} is relatively small as compared to the buffer size, it is a realistic assumption. Next, we ignore the dynamics of c_n in the RED algorithm and compute $p(\widehat{q})$ as the probability of discarding a packet when $\widehat{q}_n = \widehat{q}$, by taking $c_n = 0$. This simplifies the analysis and as our simulation results will indicate, does not affect the performance much. In the rest of the paper also, except for the stability result, we will ignore the c_n dynamics. When $\widehat{q} < T_{min}$, there is no packet loss. Since $\widehat{q}_n$ is slowly changing, if $\widehat{q}_o$ is small compared to T_{min}, $\{\widehat{q}_n\}$ will spend a significant amount of time below T_{min}. Thus, we first study the system dynamics in this region.

We will assume that the maximum TCP window size W_{max} allowed by the receiver is large enough such that the TCP source can fill up the propagation pipe if needed (actually when there is a packet loss, a rough condition for this is $E[W] > E[\Delta](1-\rho)/E[s_T]$ where $E[W]$ is the mean window size when the queue drops packets with a certain probability, $\Delta = \Delta_1 + \Delta_2$ and ρ is the exogenous traffic intensity). In fact this is recommended in practice so that link utilization is good enough. When $\widehat{q}_n < T_{min}$, after some initial change, the window size will stay constant at W_{max}. Let $E_\pi[q]$, $E_\pi[q^T]$ denote the mean queue length and the mean number of TCP packets in the queue under the (quasi)stationary distribution. Let $E_\pi[V_T]$ be the mean sojourn time of the TCP packets in the queue and λ_T the TCP goodput. Then by applying Little's law to the TCP stream of packets in the queue,

$$\lambda_T E_\pi[V_T] = E_\pi[q^T]. \tag{8}$$

Similarly by applying Little's law on the propagation pipe (in the backward and the forward direction) and observing that the rate at which ACKs are generated from the receiver is same as λ_T (since there is no packet loss),

$$\lambda_T E[\Delta] = W_{max} - E_\pi[q^T]. \tag{9}$$

Furthermore, the window size being constant, if the exogenous traffic is Poisson, and we assume FIFO discipline, for $\Delta = 0$ we are in the framework of Boxma and Cohen[8] and Sharma and Mazumdar[36]. Since the result is central to our approach, we have provided it in the generality of multiple TCPs in appendix A. Then (see e.g. [8], (2.14) and appendix A)

$$\begin{aligned} E_\pi[q] &= (1+E[Y])W_{max}, \\ E_\pi[V_T] &= E_\pi[q_T]E[s_T] + \lambda E_\pi[q^T]E[s_T]\frac{E[s]}{1-\lambda E[s]} \end{aligned} \tag{10}$$

where Y is the number of packets served in a busy period of an M/G/1 queue with Poisson arrivals with rate λ, service times i.i.d. with the distribution of the exogenous packet lengths and the initial workload in the queue equal to that of one TCP packet. Thus

$$E[Y] = \frac{\lambda E[s_T]}{1-\lambda E[s]}. \tag{11}$$

Solving (7)-(10) we obtain λ_T, $E_\pi[q^T]$, $E_\pi[q]$ and $E_\pi[V_T]$. Explicilty, $\lambda_T = (1 - \lambda E[s])/E[s_T]$. Even though $E_\pi[q]$ obtained in (10) is the mean queue length seen by an arriving TCP packet, we take this as an approximation for mean time stationary queue length also.

When $\Delta > 0$ (with a positive probability), we use the expression in (10) to compute $E_\pi[q]$ but replace W_{max} with $E_\pi[q^T]$. This approximation is reasonable in view of the analysis in Sharma and Mazumdar[36]. The intuition behind it is that if W_{max} is reasonably large then there will be enough ACKs and TCP packets propagating such that within a short time of a TCP packet leaving the queue, a TCP packet enters the queue. Thus, we could think of the system without propagation delays but with fixed window size equal to $E_\pi[q^T]$. We will see in section 4 that the simulations support this idea.

Next consider the case when $\widehat{q} > T_{min}$. Then the probability of discarding a packet is $p(\widehat{q})$. The packets are discarded independently of each other. Thus, the process $\{(w_k, h_k)\}$ now depends only on $p(\widehat{q})$ and is independent of the queue dynamics as long as the TCP timers do not timeout very frequently. Thus, using the results of Padhye et al.[31], we obtain the mean window size $E[W] = min(W_{max}, 1 + \sqrt{}(8(1-p(\widehat{q}))/3p(\widehat{q}) + 1))$. Replacing W_{max} with $E[W]$ in (9), (10) and $\lambda E[s_T]$ with $1 - \rho(1 - p(\widehat{q}))$ we obtain approximations for $E[V_T]$ and $E_\pi[q|\widehat{q}]$ when $\widehat{q} > T_{min}$ for both $\Delta = 0$ and for $\Delta > 0$.

We have used the approximation that the TCP packets can fill the propagation pipe. We will see in section 4 that if this assumption is satisfied, the quasistationary moments obtained above provide acceptable accuracy and in fact the ODE obtained via them is also quite accurate.

If the exogenous arrival stream is an MMPP we use the above equations but replace the external arrival rate λ by $E_\pi[\lambda]$, the time average arrival rate for the MMPP stream. Of course Little's law still holds but, unlike for the Poisson stream, we do not have a rigorous justification for (10). However a proof of (10) for MMPP can be carried out if we ignore certain dependencies introduced by the Markov modulated arrival rates. In the section on simulations we will see that the approximation works well for some cases.

2.4 Multiple TCP connections

In this section we extend all the results provided so far for the system with one TCP connection to the system in which N TCP connections pass through the queue (see Fig.

4). TCP packets of user i experience a total propagation delay of Δ_i. The process (w_k, h_k, q_k^T) for user i will be denoted by $(w_k(i), h_k(i), q_k^T(i))$. The maximum window size for connection i is $W_{max}(i)$. A generic service time of packets of connection i will be denoted by $s_T(i)$ while that of the exogenous stream will be denoted by s. Our presentation of stability arguments as well as of performance analysis will be brief and we only point out the changes required in the arguments for the single TCP case.

First we consider the stability of the system. When the buffer size is infinite, the stability of the system can be shown as in the one TCP case. When the buffer size is finite (B bits), we illustrate the proof for the case when two TCP connections share a bottleneck queue. Then the state of the system will be $(w_t(1), v_t, h_t(1), q_t^T(1), w_t(2), h_t(2), q_t^T(2))$. Let t_k be the time when the kth updation of a TCP source window size (of any of the two sources) takes place, and let $(w_k(1), v_k, h_k(1), q_k^T(1), w_k(2), h_k(2), q_k^T(2))$ be the state of the system at $t = t_k^+$. We assume that the total propagation delay $\Delta_i = \Delta_{i1} + \Delta_{i2}$ for $i = 1, 2$, and where $\Delta_{ij} = \Delta'_{ij} + \Delta''_{ij}$, for $i = 1, 2$, $j = 1, 2$. Δ'_{ij} is the deterministic component of the delay and Δ''_{ij} is the random (queueing at other nonbottleneck queues and processing delays) component of the delay. Δ''_{ij}, for $j = 1, 2$ are assumed to have density functions. Then, between any two consecutive TCP packets (of either source) arriving at the queue, there is a time gap of $\delta > 0$ with probability $p_T > 0$. We also make assumptions for the system as in Theorems 1, 2. Then, using arguments exactly as in Theorems 1 and 2 (for TCP Tahoe and Reno), a tagged packet from each of the two TCP sources can be successively dropped on arrival at the queue so that starting from any arbitrary state $(w_1, v, h_1, q_1, w_2, h_2, q_2)$ we can reach the state $(1, 0, 2, 0, 1, 2, 0)$ in a finite upper bounded time with a positive probability (lower bounded away from zero). This ensures the boundedness of moments of the interregeneration time of the process $(w_k(1), v_k, h_k(1), q_k^T(1), w_k(2), h_k(2), q_k^T(2))$ to the state $(1, 0, 2, 0, 1, 2, 0)$. Also, as in section 2.2.2, we can show the stability of the system with two TCP connections sharing a RED controlled queue with an exogenous stream.

Next, we consider the (quasi)stationary stability of the system with RED control. Again, we consider the case when there are two TCP connections. When $\widehat{q}_n \equiv \widehat{q} \leq T_{min}$, no packet is dropped. Thus all TCP windows sizes increase to the maximum lengths and stay there. Then from appendix A, the (quasi)stationary stability of the system can be obtained. When $\widehat{q}_n \equiv \widehat{q} > T_{min}$, every packet (TCP or UDP) entering the queue gets dropped with probability $p(\widehat{q})$. Visits to the state $(1, 0, 2, 0, 1, 2, 0)$, with no TCP packets/ACKs propagating and a packet leaving a TCP source, serve as regeneration epochs for the system. Using arguments very similar to those in Theorems 1 and 2 (and hence not reproduced here), we can show that the processes $(w_t(1), v_t, h_t(1), q_t^T(1), w_t(2), h_t(2), q_t^T(2))$ and $(w_k(1), v_k, h_k(1), q_k^T(1), w_k(2), h_k(2), q_k^T(2))$ have unique (quasi) stationary distributions (for both TCP Tahoe and Reno). We do not need the assumptions (1) - (3), ((1), (2′), (3)) of the two theorems in the proofs.

Next we consider the ODE approximation; specifically the convergence of the process $\widehat{z}_t^\beta$ to z_t. The proof supplied in appendix B for the single TCP case holds with the following minor changes. Now the state space of the process should indicate for each packet in the queue the stream to which it belongs. Similarly one should indicate the stream of each packet arriving to the queue. For a fixed $\widehat{q}$ the process $\{\xi_n^\epsilon\}$ (defined in appendix B) so obtained is again an irreducible, finite state Markov chain. Thus all the arguments as for the case of the single TCP connection hold verbatim. Also it will become obvious from

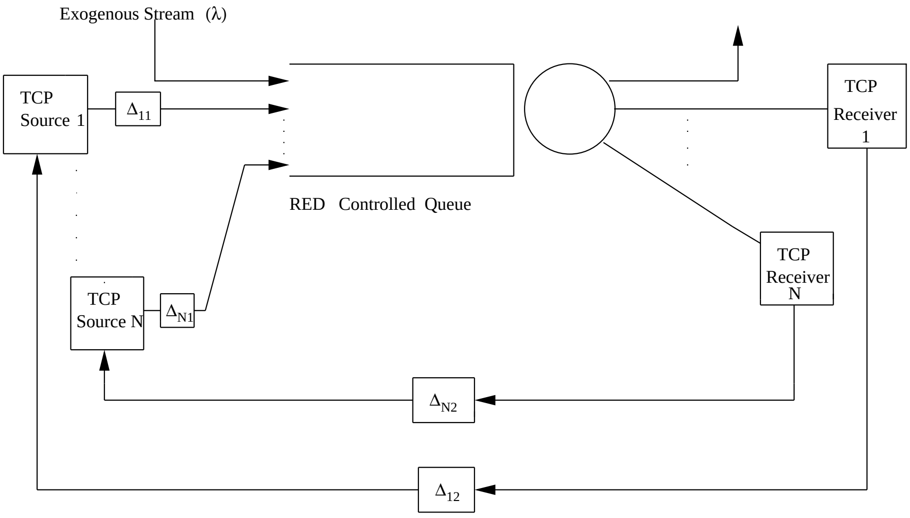

Figure 4: The model for the Multi-TCP case

the discussion below, that the arguments for the existence and uniqueness of the solution of the ODE and uniqueness of the global attractor remain the same.

Next we obtain the quasistationary moments for the system. First consider the case when $\Delta = 0$ and $\widehat{q} < T_{min}$. Then, from Appendix A, for constant window sizes we get, approximately,

$$E_\pi[q|\widehat{q}] = \sum_{i=1}^{N} (1 + E[Y(i)])W_{max}(i) \tag{12}$$

where $Y(i)$ is the number of packets served in a busy period in an M/GI/1 queue with the arrival rate λ, the service times i.i.d. with the distribution of s and the initial workload of $s_T(i)$. Therefore,

$$E[Y(i)] = \frac{\lambda E[s_T(i)]}{1 - \lambda E[s]}. \tag{13}$$

Also the mean sojourn time of packets of the ith TCP connection is

$$E_\pi[V_T(i)] = W_{max}(i)E[s_T(i)] + \sum_{j \neq i} W_{max}(j)E[s_T(j)] + \lambda \sum_{j} \frac{W_{max}(j)E[(s_T(j)]}{1 - \lambda E[s]} E[s]. \tag{14}$$

The approximation involved in (12) is that, we may require time stationary mean $E_\pi[q|\widehat{q}]$ while (12) is the exact mean queue length seen by a TCP packet arriving at the queue.

Using Little's law for individual connections separately,

$$\lambda_T(i) = \frac{W_{max}(i)}{E[V_T(i)]}. \tag{15}$$

From (12), (13) and (14) we obtain $E_\pi[q|\widehat{q}]$, $E_\pi[V_T(i)]$ and $\lambda_T(i)$, the main performance indices. Of course, now the mean waiting time of the exogenous stream can also be obtained by applying Little's law on the overall traffic.

For $\widehat{q} > T_{min}$, $\Delta = 0$, replace $W_{max}(i)$ in the above equations by $E[W(i)]$, obtained for TCP connection i via Padhye's approximation (applied to the ith connection with probability of packet loss $p(\widehat{q})$.

Next consider the case of $\Delta > 0$, $\widehat{q} < T_{min}$. Now some of the TCP packets/ACKs can be propagating in the links. Using Little's law for each connection on the queue,

$$\lambda_T(i)E[V_T(i)] = E[q^T(i)]. \tag{16}$$

Again using Little's law for each connection on the packets/ACKs propagating in the pipe,

$$\lambda_T(i)E[\Delta_i] = W_{max}(i) - E[q^T(i)]. \tag{17}$$

We further use the approximation (based on (12) and (13))

$$E_\pi[q|\widehat{q}] = \textstyle\sum_{i=1}^N (1 + E[Y(i)])E[q^T(i)].$$

and corresponding to (14) where $W_{max}(i)$ is replaced by $E[q^T(i)]$ to obtain

$$E_\pi[V_T(i)] = \frac{\sum_j E[q^T(j)]E(s_T(j)]}{1-\rho} \tag{18}$$

These equations can be solved to obtain $E_\pi[q|\widehat{q}]$, $E_\pi[V_T(i)]$ and $\lambda_T(i)$ and then we can also obtain the mean waiting time of the exogenous stream as before.

When $\Delta > 0$, $\widehat{q} \geq T_{min}$, we should replace $W_{max}(i)$ in (16)-(18) by $E[W(i)]$, using Padhye's approximation. Now one more change is required because the TCP throughput and goodput are not the same anymore. Letting $\lambda'_T(i)$ denote the goodput, if $p(\widehat{q})$ is the probabilty of loss, $\lambda'_T(i) = \lambda_T(i)(1 - p(\widehat{q}))$. We need to use $\lambda'_T(i)$ instead of $\lambda_T(i)$ in (16) (since we have assumed propagation delay only in the reverse direction). If we have Δ_1 propagation delay in the forward direction (i.e, before the queue) and Δ_2 in the reverse direction then we will need Little's law in both the propagation pipes (if $p(\widehat{q})$ is not very small) and use $\lambda_T(i)$ in the forward direction and $\lambda'_T(i)$ in the reverse direction.

3 The multiple router case

In this section we extend the stability and performance analysis results of section 2 to the case where the TCP connections pass through several routers. The queues on these routers will have exogenous traffic also. The routers employ RED control. This is a more typical situation in a network. However the analysis becomes harder. Thus we will rely more on approximations in this section. For those approximations we will provide the intuition and finally we will verify their accuracy through the simulations provided in section 4. For a system where a TCP connection passes through a tandem of RED controlled queues, we present the stability analysis in section 3.1 and the performance analysis in section 3.2. Our approach can be extended to the system with multiple TCP connections passing through multiple routers. However just as in usual queueing systems, one needs to analyse each system separately. As an example, in section 3.3 we present a particular case study. In all the cases considered we will provide our analysis for the two queue case. Extension of the ideas for more than two queues will be obvious.

Consider the system shown in Figure 5. The exogenous stream to Q_i, $i = 1, 2$, is a Poisson process with rate λ_i. The exogenous stream i enters Q_i and leaves the system on departure from the queue. The TCP stream enters Q_1 and on departure from it enters

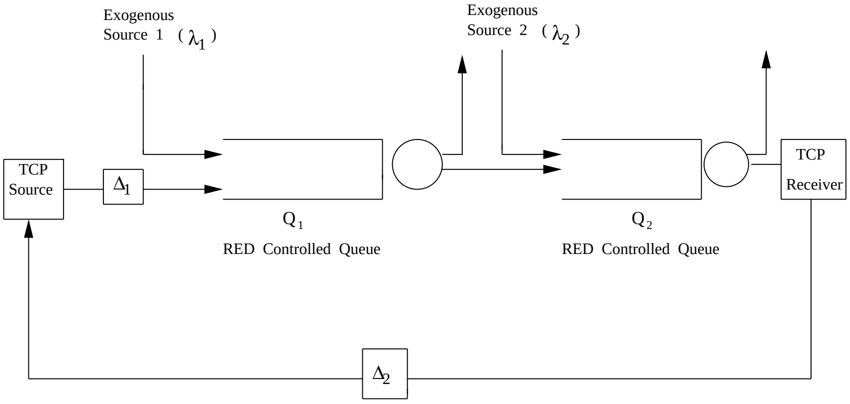

Figure 5: The tandem queues model

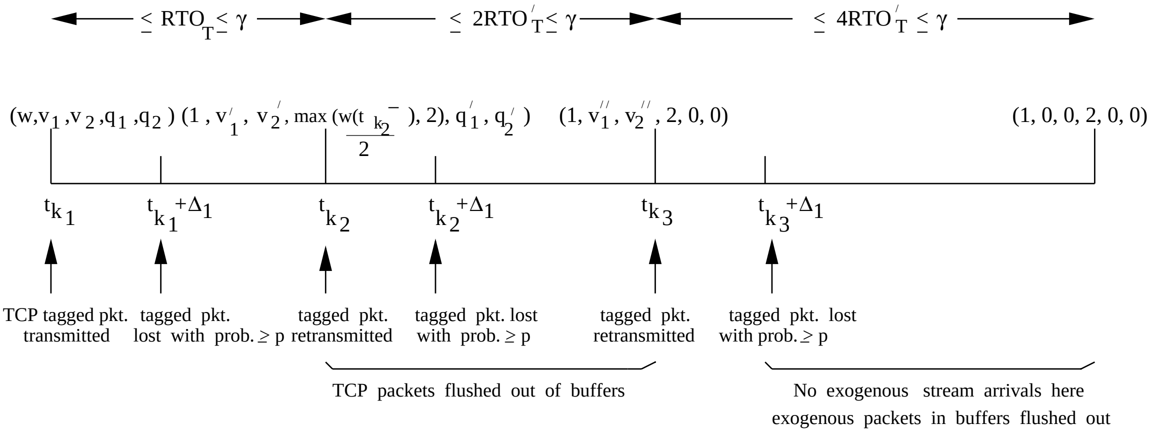

Figure 6: Diagram illustrating the sequence of states in stability proof of Theorem 3

Q_2. On successful transmission through Q_2, an ACK is sent back to the TCP source. The delay of the path is $\Delta = \Delta_1 + \Delta_2$, where, as in section 2, both Δ_1 and Δ_2 have deterministic components Δ_1' and Δ_2' and random queueing delay components Δ_1'' and Δ_2''. Also, as in section 2, because of the queueing at intermediate nodes, the interarrival time at the first queue between two consecutive TCP packets $> \delta$ (where δ is a small positive constant) with a positive probability p_T. Each queue employs RED control. The processes $v_k, q_k^T, c_k, \widehat{q}_k$ corresponding to Q_i will be denoted by $v_k(i), q_k^T(i), c_k(i), \widehat{q}_k(i)$. The generic service time of a TCP packet is denoted by s_T whereas the service times of a packet of the exogenous sources 1 and 2 are respectively $s(1)$ and $s(2)$. The assumptions on the retransmission timeout period (RTO) of the TCP source are exactly the same as for the single queue case.

3.1 Stability Analysis

Theorem 3 (Stability for the TCP Tahoe case) *Assume*

(1) *the distribution function of* $s(1)$ *is absolutely continuous in* $[0, \epsilon]$ *for some fixed* $\epsilon > 0$,

(2) $B_1 + B_2 < 2\Delta'$ *where* $\Delta' = \Delta_1' + \Delta_2'$,

(3) *$P[s_T \geq \alpha_{min}] > 0$ for some $\alpha_{min} > 0$.*

Then the processes $\{(w_k, v_k(1), v_k(2), h_k, q_k^T(1), q_k^T(2))\}$ and $\{(w_t, v_t(1), v_t(2), h_t, q_t^T(1), q_t^T(2))\}$ have unique stationary distributions. Also, starting from any arbitrary initial state, processes $\{(w_k, v_k(1), v_k(2), h_k, q_k^T(1), q_k^T(2))\}$ and $\{(w_t, v_t(1), v_t(2), h_t, q_t^T(1), q_t^T(2))\}$ converge to their stationary distributions in total variation.

Proof. Whenever $(w_k, v_{1k}, v_{2k}, h_k, q_k^T(1), q_k^T(2)) = (1, 0, 0, 2, 0, 0)$ the system can be considered to regenerate. Thus if we show that the intervisit time τ to this state satisfies $E[\tau] < \infty$, by regenerative process theory the process has a unique stationary distribution π. Also if we show that τ has an aperiodic distribution then this process converges to π in total variation from any arbitrary initial state.

We start with showing $E[\tau^\beta] < \infty$ for any $\beta > 0$. By the geometric coin tossing argument, it is sufficient to show that starting in any arbitrary initial state $(w, v_1, v_2, h, q_1, q_2)$ in a finite (upper bounded) time the state $(1, 0, 0, 2, 0, 0)$ can be reached with a positive probability (bounded away from zero).

We consider an epoch in the system evolution, t_{k_1}, when the system is in state $(w, v_1, v_2, h, q_1, q_2)$ ($w \geq 1$, $v_1 \geq 0$, $v_2 \geq 0$, $h \geq 2$, $q_1 \geq 0$, $q_2 \geq 0$) (see Figure 6) and the source transmits a *tagged* packet (in any case within a finite time $\leq \gamma$ a packet will leave the source). Let $\delta_1 = \delta/3$.

Reasoning as in the proof of Theorem 1 (with only notational changes) with probability $\geq p$ the tagged packet will be lost at the first queue. Thus within a finite time (upper bounded by $RTO_T \leq \gamma$) by which 3 duplicate ACKs are received at the source or a timeout occurs (whichever is smaller), the system state becomes at time, say t_{k_2}, $(1, v_1', v_2', max(\frac{w_{t_{k_2}^-}}{2}, 2), q_1', q_2')$ for some $v_1' \geq 0$, $v_2' \geq 0$, $q_1' \geq 0$, $q_2' \geq 0$. It is possible that $w_{t_{k_2}^-} > w$ because of successful transmissions before the tagged packet during time $[t_{k_1}, t_{k_2}]$. The tagged packet is immediately retransmitted and this packet on arrival at the queue Q_1 after time Δ_1 will again be lost with a probability $\geq p$. Thus, after time $2RTO_T' \leq \gamma$ (timeout occurs), the system goes to state $(1, v_1'', v_2'', 2, 0, 0)$ (at time, say t_{k_3}) from state $(1, v_1', v_2', max(\frac{w(t_{k_2}^-)}{2}, 2), q_1', q_2')$, for some $v_1'' \geq 0$ and $v_2'' \geq 0$, with probability $\geq p$ (TCP packets get flushed out of the buffers because of assumption (2) and because $RTO_T' \geq \Delta'$). The tagged packet is again retransmitted (RTO is now $4RTO_T'$ (exponential backoff)) and with a probability $\geq p$ is lost. Also with probability $e^{-\lambda_1(4RTO_T' - \Delta_1)}.e^{-\lambda_2(4RTO_T' - \Delta_1)} \geq e^{-\lambda_1(\gamma - \Delta_1')}.e^{-\lambda_2(\gamma - \Delta_1')}$ no exogenous stream arrivals occur in time $4RTO_T' - \Delta_1$, and then using assumption (2), after time $4RTO_T' \leq \gamma$ we reach the state $(1, 0, 0, 2, 0, 0)$ from the state $(1, v_1'', v_2'', 2, 0, 0)$ with probability $\geq e^{-(\lambda_1+\lambda_2)(\gamma - \Delta_1')}p$. Thus we have shown that starting from any initial state $(w, v_1, v_2, h, q_1, q_2)$ in a finite upper bounded time we can obtain a regeneration with a positive (bounded away from zero) probability. Therefore, as mentioned above, by a geometric coin tossing argument, we obtain $E[e^{\theta\tau}] < \infty$ for some $\theta > 0$.

The aperiodicity of distribution of τ is obtained as in Theorem 1. Similarly, as in Theorem 1, we can show that for the continuous time process, the regeneration length $\hat{\tau}$ satisfies $E[e^{\widehat{\theta\tau}}] < \infty$ for some $\theta > 0$ and $\hat{\tau}$ has a spread-out distribution. □

The rate of convergence to the stationary distribution can be obtained as in the one queue case.

As in Theorem 3, modifying the proof of Theorem 2, we obtain

Theorem 4 (Stability for the TCP Reno case) *Under assumptions* (1) *and* (3) *as for the TCP Tahoe case and the assumption* (2) $B_1 + B_2 < \Delta'$ *where* $\Delta' = \Delta'_1 + \Delta'_2$ *the processes* $\{(w_k, v_k(1), v_k(2), h_k,\ q^T_k(1), q^T_k(2))\}$ *and* $\{(w_t, v_t(1), v_t(2), h_t,\ q^T_t(1), q^T_t(2))\}$ *have unique stationary distribution. Also, the processes* $\{(w_k, v_k(1), v_k(2), h_k, q^T_k(1), q^T_k(2))\}$ *and* $\{(w_t, v_t(1),\ v_t(1), h_t, q^T_t(1), q^T_t(2))\}$ *converge to* π *in total variation, starting from any arbitrary initial state.* □

The rates of convergence for the system in Theorem 4 are also obtained as before.

For the system in Figure 5 with the (finite buffer) queues employing RED control we consider the process $\{(w_k, v_k(1), v_k(2), h_k, q^T_k(1), q^T_k(2), \widehat{q}_k(1), \widehat{q}_k(2), c_k(1), c_k(2))\}$ at times t_k. We consider the intervisit times to state $(1, 0, 0, 2, 0, 0, 0, 0, -1, -1)$ when a TCP packet just leaves the source and there are no TCP packets/ACKs propagating in the system. These serve as regeneration epochs for the process $\{(w_k, v_k(1), v_k(2), h_k, q^T_k(1), q^T_k(2), \widehat{q}_k(1), \widehat{q}_k(2), c_k(1), c_k(2))\}$. Following exactly similar lines of reasoning as in section 2.2.2, with only notational changes, we can show the finiteness of the moments of the intervisit times and thus we can show the stability of the system in Figure 5 with RED controlled queues.

3.2 Performance analysis

For performance analysis we follow the approach of the previous sections. Define processes $\{\widehat{z}^\beta_t(1), t \geq 0\}$ and $\{\widehat{z}^\beta_t(2), t \geq 0\}$ as,

$$\widehat{z}^\beta_t(i) = \widehat{q}_{\lfloor \frac{t}{\beta} \rfloor}(i), \qquad \widehat{z}^\beta_0(i) = \widehat{q}_o(i), \qquad i = 1, 2. \tag{19}$$

Consider the system of two ODEs

$$\frac{dz_t(i)}{dt} = E_\pi[q(i)|z_t(1), z_t(2)] - z_t(i), \qquad z_0(i) = \widehat{q}_0(i), \qquad i = 1, 2. \tag{20}$$

We provide the necessary changes required in the arguments for the single queue system to prove that as $\beta \downarrow 0$ for any $0 < T < \infty$,

$$\sup_{0 \leq t \leq T} \left| \widehat{z}^\beta_t(i) - z_t(i) \right| \xrightarrow{P} 0, \qquad i = 1, 2.$$

To carry out the proof for the above convergence, we make one small modification in the system. We update the $\widehat{q}_n(1)$ and $\widehat{q}_n(2)$ at the arrival epochs to both the queues. This changes the dynamics of the system in a very minor way but is convenient in the proofs. Then we consider the process (in the notation of the appendix B) $\{\xi^\epsilon_n\}$ which includes in the state space the type of the packet at each position in the queue and also the indication as to which queue the packet arrives at, at the nth arrival epoch. Also, let $\theta^\epsilon_n = (\widehat{q}^\beta_n(1), \widehat{q}^\beta_n(2))$. Under the exponential distribution assumptions in appendix B, for a fixed $\theta^\epsilon_n = \theta$, $\{\xi^\epsilon_n\}$ is still an irreducible, finite state Markov chain. The continuity requirements continue to hold. At the end of this section we will comment on the existence and the uniqueness of the solutions of the coupled ODE (20). We will also provide asymptotic results on the behavior of the solutions of the ODE and on the approximation of stationary distributions of $\widehat{q}_n^{\beta}$.

To obtain the needed performance parameters and to solve the ODE (20), we need certain quasistationary moments which we now provide. First consider the case when $\Delta = 0$ a.s. and $\widehat{q}(i) < T_{min}(i),\ i = 1, 2$. Then

$$E_{\pi}[q^T(1)] + E_{\pi}[q^T(2)] = W_{max}. \tag{21}$$

Applying Little's law at each queue for the TCP packets,

$$\lambda_T E_{\pi}[V_T(i)] = E_{\pi}[q^T(i)], \qquad i = 1, 2. \tag{22}$$

Now we use the approximation

$$E_{\pi}[q(i)] = E_{\pi}[q^T(i)](1 + E[Y_i]), \qquad i = 1, 2. \tag{23}$$

The mean sojourn time $E_{\pi}[V_T(i)]$ of the TCP packets in Q_i is

$$\begin{aligned} E_{\pi}[V_T(i)] &= E_{\pi}[q^T(i)]E[s_T] + \frac{\lambda_i E_{\pi}[q^T(i)]E[s_T]E[s_T(i)]}{1 - \lambda_i E[s_T(i)]} \\ &= \frac{E_{\pi}[q^T(i)]E[s_T]}{1 - \rho_i}, \qquad i = 1, 2, \end{aligned} \tag{24}$$

where s_T denotes a generic service time of a TCP packet and $s_T(i)$ denotes a generic service time of the exogenous stream coming into the ith queue and $\rho_i = \lambda_i E[s_T(i)]$.

From equations (21)-(24) we can solve for $E_{\pi}[q^T(i)]$, $E_{\pi}[V_T(i)]$, $E_{\pi}[q(i)]$ for $i = 1, 2$, and for λ_T as before. However if $\rho_1 = \rho_2$ and the queues never become empty (the case usually considered in this paper) then the two equations in (22) and (24) become the same because $\lambda_T = (1 - \rho_i)/E[s_T]$. Therefore, we do not have enough equations to solve for all the unknowns. In that case, since this becomes a symmetric case, we take $E[q^T(1)] = E[q^T(2)]$. Thus from $\lambda_T = (1 - \rho_i)/E[s_T]$, now we can solve for all the unknowns. If $\rho_1 \neq \rho_2$ then (say) $\rho_1 > \rho_2$ and hence Q_1 becomes the bottleneck queue, we take $E[q^T(1)] = W_{max} - 1$ and $E[q^T(2)] = 1$ (this is the more likely event to happen than $E[q^T(1)] = W_{max}$ and $E[q^T(2)] = 0$). Our simulations in section 4 will verify these heuristics.

The changes needed to take care of $\Delta > 0$ (with positive probability) and $\widehat{q}(i) \geq T_{min}(i)$ for either of $i = 1, 2$ are as before.

Now we discuss the ODE (20). The Lipschitz continuity of $E_{\pi}[q(i)|z_t(1), z_t(2)],\ i = 1, 2$ can be carried out, as for the single queue case, by showing the continuous differentiability of this as a function of $z_t(1), z_t(1)$ (since $z_1,\ z_2$ are bounded). Thus the ODE has a unique solution for each initial condition. By boundedness of $z_t(1),\ z_t(2)$, the solution of the ODE exists for all time. Also, by Brouwer's fixed point theorem there is an equlibrium point (z_1^*, z_2^*) of (20).

Now we study the asymptotic behavior of the solutions of the ODE and of the stationary distributions of $\widehat{q}_n^{\beta}$. We can show using the methods in [16], Prop. 2.1 that

$$\sup_{0 \leq t \leq T} \ \sup_{\widehat{q}_0(j) \leq T_{max}(j), j=1,2} \left| \widehat{z}_t^{\beta}(i) - z_t(i) \right| \xrightarrow{P} 0, \qquad i = 1, 2.$$

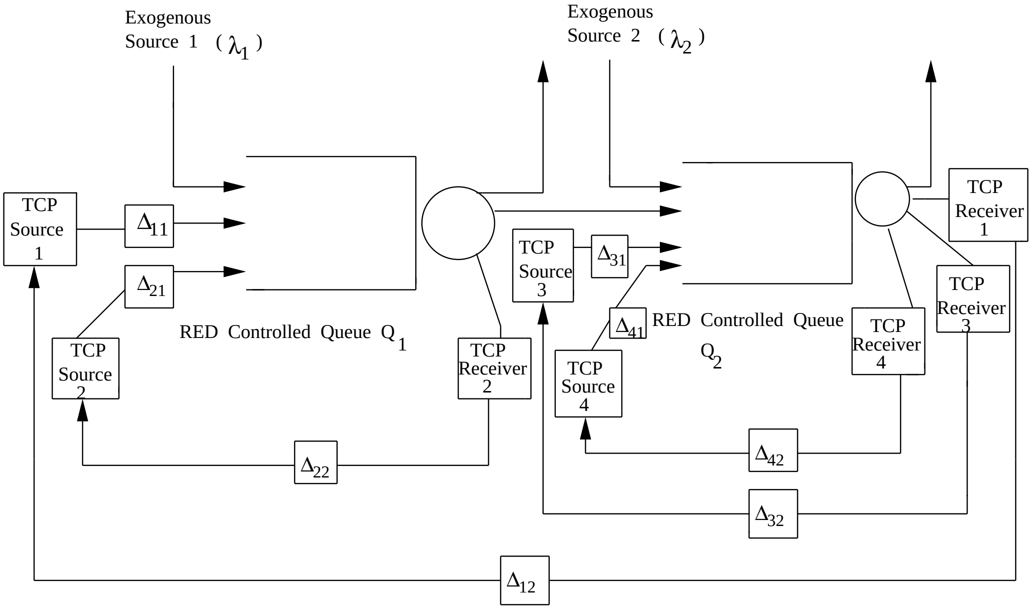

Figure 7: Multiple TCP connections through multiple RED queues

Then the proof of Theorem 2.2 in [16] can be carried out in the same way for our setup (because we have already proved the existence and uniqueness of stationary distributions π^β of $(\widehat{q}_t^\beta(1), \widehat{q}_t^\beta(2))$ for each β and since these processes are bounded, their stationary sequences form a tight sequence). Therefore, any weak limit π^0 of π^β as $\beta \downarrow 0$, is an invariant distribution of ODE (20). Next consider the Bendixson criterion [15]. The $E[q_i|z_1, z_2], i = 1, 2$ have been shown to be continuously differentiable. Also, the divergence of right side of ODE (20)

$$(\frac{\partial}{\partial z_1} E[q_1|z_1, z_2] - 1) + (\frac{\partial}{\partial z_2} E[q_2|z_1, z_2] - 1)$$

is never zero because $\frac{\partial}{\partial z_i} E[q_i|z_1, z_2] \leq 0$ for $i = 1, 2$. Therefore, from boundedness of $\widehat{q}_i^\beta, i = 1, 2$, the ODE has no cycles or pseudocycles. Now, using the Poincare-Bendixson theorem [15], the ODE always converges to an equilibrium point. Furthermore, by Prop. 2.6 in [16], the support of the weak limit π^0 is the set of equilibrium points of the ODE. From section 3.1.1 of [16], we also know that if q^* is an equilibrium point of the ODE (20) such that all eigenvalues of the derivative of RHS of (20) at q^* have positive real parts, (then q^* is a repeller) then $\pi^0(q^*) = 0$. However, in appendix C we show that all equilibrium points of (20) are local attractors.

3.3 Multiple TCP connections through multiple RED queues

Let's consider the system shown in Figure 7. In this system there are multiple TCP connections passing through a tandem of two RED controlled queues. The first queue Q_1 has two TCP connections passing through it. The second queue Q_2 has three TCP connections passing through it. TCP connection 1 traverses both the routers. The total propagation delay for TCP connection i is $\Delta_i = \Delta_{i1} + \Delta_{i2}$. Each queue also has an exogenous arrival stream. We carry out performance analysis of this system and indicate how the approach initiated here could probably be generalised to take care of more complicated systems. We note that for most cases of interest our approximations work well and provide accurate estimates of the performance indices of the system. The simulations presented in section 4 verify our approximations.

We do not carry out the stability analysis for this system. This can (probably) be carried out as before but will require a specific proof. Since each such system will require its own proof, just like in usual queueing systems, we do not elaborate. However the approximate formulae for performance analysis have a more common scheme which we carry out in the following. From a more practical point of view, these are anyway more important. We again follow the decomposition approach used above to obtain the solution of the ODE (20) for the system, for which we again need the quasistationary moments for the system. Let $E_\pi[V_T(i)]$ and $E_\pi[q^T(i)]$ denote the mean sojourn time and the mean number of TCP i packets in the (respective) queues for $i = 2, 3, 4$, under quasistationarity. For TCP connection 1, which passes through both the queues, let $E_\pi[V_T(1,1)]$ ($E_\pi[V_T(1,2)]$) and $E_\pi[q^T(1,1)]$ ($E_\pi[q^T(1,2)]$) denote the mean sojourn time and mean number of TCP 1 packets in queue 1 (2) respectively. The maximum window size and throughput of TCP connection i are respectively $W_{max}(i)$ and $\lambda_T(i)$. The service time of a TCP connection i packet is denoted by $s_T(i)$. The traffic intensity of the Poisson stream into queue Q_i is denoted by $\rho_i = \lambda_i E[s_T(i)]$ for $i = 1, 2$. Then for $\Delta_i \geq 0$, $\widehat{q}(i) < T_{min}(i)$, by applying Little's law for each of the TCP streams in the queues and in the propagation pipes we obtain the following set of equations

$$\begin{aligned} W_{max}(1) - \lambda_T(1)E[\Delta_1] &= E_\pi[q^T(1,1)] + E_\pi[q^T(1,2)], && (25) \\ W_{max}(i) - \lambda_T(i)E[\Delta_i] &= E_\pi[q^T(i)], \quad i = 2,3,4, && (26) \\ \lambda_T(1)E_\pi[V_T(1,1)] &= E_\pi[q^T(1,1)], && (27) \\ \lambda_T(1)E_\pi[V_T(1,2)] &= E_\pi[q^T(1,2)], && (28) \\ \lambda_T(i)E_\pi[V_T(i)] &= E_\pi[q^T(i)], \quad i = 2,3,4. && (29) \end{aligned}$$

Further using approximations as before (see section 2) we obtain for the mean sojourn times of TCP packets in the two queues

$$\begin{aligned} E_\pi[V_T(1,1)] &= \frac{E_\pi[q^T(1,1)]E[s_T(1)] + E_\pi[q^T(2)]E[s_T(2)]}{1-\rho_1}, && (30) \\ E_\pi[V_T(2)] &= \frac{E_\pi[q^T(1,1)]E[s_T(1)] + E_\pi[q^T(2)]E[s_T(2)]}{1-\rho_1}, && (31) \\ E_\pi[V_T(1,2)] &= \frac{E_\pi[q^T(1,2)]E[s_T(1)] + E_\pi[q^T(3)]E[s_T(3)] + E_\pi[q^T(4)]E[s_T(4)]}{1-\rho_2}, && (32) \\ E_\pi[V_T(3)] &= \frac{E_\pi[q^T(1,2)]E[s_T(1)] + E_\pi[q^T(3)]E[s_T(3)] + E_\pi[q^T(4)]E[s_T(4)]}{1-\rho_2}, && (33) \\ E_\pi[V_T(4)] &= \frac{E_\pi[q^T(1,2)]E[s_T(1)] + E_\pi[q^T(3)]E[s_T(3)] + E_\pi[q^T(4)]E[s_T(4)]}{1-\rho_2}. && (34) \end{aligned}$$

The above set of equations are just sufficient to solve for all the unknowns and obtain all the quasistationary moments of the system when $\widehat{q}(i) < T_{min}(i)$. When $\widehat{q}(i) \geq T_{min}(i)$ we replace the $W_{max}(i)$'s in the above equations by $E[W(p_1,p_2)(i)]$'s obtained from Padhye's approximation (p_1 and p_2 represent the probabilities of packet loss in Q_1 and Q_2

respectively) and make further relevant changes because now the incoming exogenous Poisson arrival rates into the two queues are respectively $\lambda_1(1-p_1)$ and $\lambda_2(1-p_2)$. Eventhough in Padhye's et al [31], the formula is provided for a single bottleneck queue, it can be used for more than one such queue also because window dynamics of a TCP depends only upon the probability of packet loss of the TCP connection. The quasistationary moments so obtained help us to obtain plots of the ODE (20) for both the queues and also help us to compute the system performance indices of interest. Simulations in section 4 verify all our approximations. We also discuss in that section various cases of interest (regarding which queues become bottlenecks etc.) and find that we do not need to make any additional approximations (as in the previous section). The above set of equations completely determine the behavior of the system under all conditions.

This method of analysis could be possibly extended to more complicated scenarios without requiring any additional insight into the specific system behavior. This means that we could apply the approximations of section 2 to each queue in a system of networked connection of RED controlled queues, use Little's law for each flow and then depending upon the scenario (topology) obtain a system of equations (just as in the present case) which could be reduced and solved for the unknowns. Once this solution is obtained it would be apparent as to which queues build up and which do not. Thus we need not have any special insight into the specific system behavior.

4 Simulation results

In this section we summarize the simulation results obtained for the various systems. Sections 4.1 and 4.2 respectively provide simulation results for the one - queue and multiple TCP - multiple queues systems. The queues in each case are RED controlled. These results verify the accuracy of our theory and approximations. The simulations were carried out using the *ns* simulator version 2.1*b*5 of UCB/LBNL. TCP sources in all our simulations employ the Reno version of the protocol. For all our simulations the links' speeds were kept fixed at 10Mbps and the packet sizes of both the TCP and the exogenous streams were kept fixed at 750 bytes. All the plots of $\tilde{z}^\beta(t)$ vs. t and performance indices obtained from simulations correspond to single sample paths unless otherwise stated. For most of the simulations the runtime ranged between 100 to 200 secs. The buffer sizes of the RED queues were kept large enough so that packet drops occur only due to RED queue's drop mechanism and not due to any buffer overflow.

4.1 The single queue, single TCP case

For the simulation results provided in this section the exogenous traffic is Poisson or MMPP with $\rho = 0.5$. The RED queue thresholds (in the model of Figure 1) are $T_{min} = 5$ and $T_{max} = 15$ and $p_{max} = 0.1$. The TCP parameters are $W_{max} = 30$ and $\Delta = 5$msec.

We first describe the results for the Poisson case. The ODEs and the process $\tilde{z}^\beta_t$ for $\beta = 10^{-3}$ and 10^{-4} are plotted in Figs 9 and 10. As expected oscillations are smaller for $\beta = 10^{-4}$. Tables 1 and 2 provide the performance indices for $\beta = 10^{-3}$ and 10^{-4}. The errors are larger for $\beta = 10^{-3}$ than for $\beta = 10^{-4}$. The mean sojourn time estimates are particularly very good. Also, throughput estimates are better for the transient case (when the theory is exact for $\Delta = 0$) than for the stationary case. However, the errors are almost always below 10%. For the cases when $\beta = 10^{-3}$, we actually encounter problems in estimating throughputs under transience ($\tilde{z}_t^{\beta} \leq T_{min}$), because the transient periods die out fast. In this case the tabulated values correspond to values estimated over several sample paths. Further explorations with varying Δ (20ms and 30ms) for the TCP connection were carried out for the same $W_{max} (= 30)$ and we notice that the queue stabilizes at lower stationary values with increasing Δ. We cannot predict for cases beyond $\Delta = 30$ms (for $W_{max} = 30$) because our theory holds only when the delay-bandwidth product of the TCP connection is smaller than W_{max}.

We next present results for the case when the exogenous traffic is an MMPP stream. The modulating Markov chain of the MMPP has two states - ON and OFF. During the OFF period there are no packet arrivals while during the ON period, packets arrive as a Poisson process with rate $\lambda = 10$Mbps. The ON and the OFF periods are independent sequences, both exponentially distributed with mean 0.006 sec. Figure 11 plots the ODE when $\beta = 10^{-4}$. Table 3 provides the performance indices. The match here is fairly good even though we apply the approximations of section 2. We have also done simulations for $\Delta = 10$, 20, 30 ms. Then, the approximation is not so good. This shows that we need better approximations in the MMPP case in order to be able to successfully predict performance for a wider variety of situations.

4.2 The multiple queue, multiple TCP case

This section presents results when a TCP connection shares a tandem of two RED controlled routers with exogenous streams. We also discuss the special case considered in section 3 where multiple TCP connections pass through a tandem of RED controlled queues. We first consider the simpler case of one TCP connection. Depending upon the intensity of the (Poisson) exogenous streams coming into the queues none of the links could be bottlenecks for the TCP connection (nonbottleneck case) or one of the links could be bottleneck and the other the nonbottleneck link for the TCP connection (bottleneck case).

We first present results of the former case. The traffic intensity $\rho (= 0.5)$ of the exogenous streams coming into the queues is the same. The RED parameters for both the links are fixed at $T_{min} = 5$, $T_{max} = 15$, $p_{max} = 0.1$ and $\beta = 10^{-4}$. For the TCP connection, $W_{max} = 50$ and $\Delta = 10$ms. Figures 12, 13 and Table 4 provide the ODE plots and the performance indices. We have used the approximations of section 3 in calculating the performance indices. Again we note that the throughput predictions are not very accurate for the case when there is a single TCP connection coming into the queues.

We now provide results when one of the links in the path of the TCP connection is a bottleneck for it. The RED parameters for both the links are $T_{min} = 5$, $T_{max} = 15$, $p_{max} = 0.1$ and $\beta = 10^{-4}$. Q_1 is the bottleneck link with the exogenous traffic intensity $\rho_1 = 0.5$. Q_2 is the nonbottleneck link with exogenous traffic intensity $\rho_2 = 0.4$. For the TCP connection $W_{max} = 30$ and $\Delta = 10$ms. Figures 14, 15 and Table 5 provide the ODE plots and the performance indices.

Consider now the special case described in section 3.3. For the system considered therein (see Figure 7), we present simulation results which show that our approximations continue to hold for this case also.

For all the simulation results discussed in this paragraph we only change the intensities of the Poisson streams (ρ_1 and ρ_2) coming into the two queues. The RED parameters for the two queues, Q_1 and Q_2, are kept fixed at $T_{min}(1) = 10$, $T_{max}(1) = 30$, $T_{min}(2) = 20$, $T_{max}(2) = 60$, $p_{max}(1) = p_{max}(2) = 0.1$ and $\beta_1 = \beta_2 = 10^{-4}$. For the four TCP connections the maximum window sizes are respectively $W_{max}(1) = 40$, $W_{max}(2) = 30$, $W_{max}(3) = 30$ and $W_{max}(4) = 30$, whereas the propagation delays are $\Delta_1 = 40$ms, $\Delta_2 = 40$ms, $\Delta_3 = 30$ms and $\Delta_4 = 40$ms. For the first case considered $\rho_1 = 0.6$ and $\rho_2 = 0.3$. Figures 16 and 17 provide the ODE plots whereas Table 6 provide the performance indices. The matches are fairly good. We now consider the case when $\rho_1 = 0.3$ and $\rho_2 = 0.6$. Once we solve the system of equations we observe that Q_1 doesn't build up beyond $T_{min}(1)$ and Q_2 becomes the bottleneck queue for TCP connection 1. (The heuristic we used for the nonbottleneck queue in the single TCP, bottleneck queue case (refer section 3.2) does not seem to work here; so, we present the ODE plots for Q_2 only. Anyway this is more important from a practical point of view). Figure 18 provides the ODE plot for Q_2 (which builds up) whereas Table 7 provides the performance indices. It is worthwhile to remind that for the theoretical calculations we do not make any additional assumptions as we do for the one TCP case (see sections 3 and 4). Also it's not so easy here to predict beforehand which queue builds up and which doesn't. Our theory then helps us to predict quantitatively how the system will behave.

We have carried out simulations for all the cases described in the above paragraph for the mutiple TCP, multiple queue case, and for $\beta = 10^{-3}$. The results show that we have good match between theory and simulations under stationarity though the oscillations in the simulation ODE plots increase.

Acknowledgement The referee's comments have substantially improved the presentation of the paper.

Appendix A

Fixed window size system

In this appendix we give the basic stability result and the formulae for mean sojourn time and throughput of the TCP connection provided in [8] and [36] for fixed window size systems. Since these are extensively used in this paper, we provide a proof of these results which is a generalization (to multiple TCP case) of the proof in [36]. For the following theorem, for brevity we only provide a part of the proof. This will atleast show how we obtain the required formulae. We use the notation of section 2.4. The proof is provided for the case when there is infinite buffer length, no RED control, Poisson exogenous stream and $\Delta(i) = 0$ for each i.

We use the following further notations. For any $x > 0$, let $N(x)$ denote the number of exogenous stream arrivals to the queue in time x. Let S_n be the sum of n *iid* service times of the exogenous stream. Let Y_i denote the number of arrivals in a busy period of an $M/G/1$ queue which starts with an initial workload with the distribution of $s_T(i)$, has an arrival rate λ and service times with the distribution of s. By $X_T = (r_t, T_{ti}, i \geq 1)$, we denote the vector with r_t the residual service time of a packet in service at time t and T_{ti} denotes the type of packet at position i in the queue (T_{ti} denotes whether the packet is from the exogenous stream or from which TCP connection, $T_{ti} = 0$ for i greater than the queue length.) We take $\{X_t\}$ to be a right-continuous process.

Theorem 5 *Assume* $\lambda E[s] < 1$ *and* $E[s_T(i)] < \infty$ *for all* i*. If* $s_T(i)$ *for some* i *and/or* s *has a nonlattice or lattice aperiodic distribution then the process* $\{X_t\}$ *(and the waiting time process) has a unique stationary distribution. Also, starting from any initial conditions the system converges weakly to the stationary distribution.*

Moreover under stationarity between a packet of TCP i *and the next TCP packet (of any type) there are exogenous packets with the distribution of* Y_i*.*

Proof: In [36], the proof is given in four steps. Step 1 shows that if we start the queue at $t = 0$ with no exogenous packet and a service of a TCP packet just starts, then the system converges weakly to a state with it's distribution mentioned in the last statement of the theorem. Step 2 shows that such a distribution for the queueing process is stationary at the epochs when a service of a TCP packet starts. In step 3, it is shown that starting from any initial conditions, the system will converge weakly to the above mentioned distribution. This also provides the uniqueness of the stationary distribution of the queueing process at the epochs when a TCP packet service starts. Finally, in step 4, it is shown that the process $\{X_t\}$ is regenerative and inter-regeneration epochs have a finite mean. This proves the theorem. In the following we prove the first two steps.

Step 1: Because of $\Delta(i) = 0$ for each i and no packet loss (thus constant window size, after a while) the ordering of different TCP packets in the queue remains same as at $t = 0$. Let the total number of all TCP packets in the queue be k. When a packet of a particular TCP connection completes service and leaves, another packet of that TCP enters the system immediately after this. At time $s_T(i)$ (say the first TCP packet is from the ith TCP connection), the first TCP packet leaves the system and is replaced by one at the end. During this time $N(s_T(i))$ packets of the exogenous stream enter the system and they will be in between the kth and the first (which just entered the system) TCP packets. We study the evolution (in time) of number of the exogenous stream packets between the kth and the first packet. The evolution of the exogenous stream packets between any other pair of TCP packets will be similar. This number will remain same (since we have FCFS discipline) till the kth packet is served. In the meanwhile between any other pair of TCP packets the number of the exogenous packets arriving will have the distribution of $N(s_T(j))$ if the first packet of this pair is from TCP j, (although they will be independent of each other). After the service of the kth packet, these $N(s_T(i))$ packets of the exogenous stream will be served and then the first TCP packet. After the service of the 1st packet, the number of exogenous packets between the kth and the 1st will be $N_2 = N(S_{N(s_T(i))} + s_T(i))$ (the two $s_T(i)$'s in this expression are independent of each other). The evolution continues this way after each service of the 1st packet (we denote these r.v.s by N_m), and they satisfy

$$N_m = N(S_{N_{m-1}} + s_T(i)), m \geq 2.$$

Observe that $s_T(i) \leq_{st} S_{N(s_T(i))} + s_T(i)$ ($X \leq_{st} Z$ denotes $P(X > a) \leq P(Z > a)$ for all a). Now, if X, Z are nonnegative r.v.s with $X \leq_{st} Z, X, Z$ independent of N and P_X, P_Z

are the distributions of X and Z then

$$P[N(X) > x] = \int_0^\infty P[N(y) > x] dP_X(y) \leq \int_0^\infty P[N(y) > x] dP_Z(y)$$

because $P[N(y) > x]$ is a nondecreasing function of y. Therefore, $N(s_T(i)) \leq_{st} N(S_{N(s_T(i))} + s_T(i))$. Similarly, $N_1 \leq_{st} N_2 \leq_{st} N_3 \leq_{st} \cdots$. Thus if we show that $\{N_m\}$ is tight then, $N_m \overset{W}{\to} Y$, where Y is a proper r.v. and $\overset{W}{\to}$ denotes weak convergence. From the evolution,

$$E[N_{n+1}] = \lambda E[s_T(i)] + \lambda^2 E[s_T(i)]E[s] + \cdots + \lambda^{n+1} E[s_T(i)]E[s]^n$$

and hence $E[N_{n+1}] \to \lambda E[s_T(i)]/(1 - \lambda E[s]) < \infty$ whenever $\lambda E[s] < 1$. This proves the tightness of $\{N_n\}$. Also, comparing the evolution of N_n with a busy period in an $M/GI/1$ queue with initial work load $s_T(i)$ we can conclude that N_n is actually, converging to the number of arrivals in a busy period of the stated $M/GI/1$ queue because

$$Y_i = N(s_T(i)) + N(S_{N(s_T(i))}) + \cdots = N(s_T(i) + S_{N(s_T(i))} + \cdots). \tag{35}$$

Step 2: Let us start the system at $t = 0$ the with the service of a TCP packet just starting (say of kth packet) and between the kth and the *1st* (and also between any other pair of TCP packets) let the number of exogenous packets be with the distribution of Y_i (i denoting the TCP connection of the first packet of the pair). After service of the kth and the *1st* packet, there will be $N(s_T(i)) + N(S_{Y_i})$ exogenous packets in between them. A moment's reflection on the working of an $M/GI/1$ queue starting with work load $s_T(i)$ will show that $N(s_T(i)) + N(S_{Y_i}) \overset{dist}{=} Y_i$ (can be observed from (35) also). Thus, we see that if we start with the distribution of the queueing process, as mentioned, in between any two TCP packets (say the first of these packets is from jth TCP connection) the number of exogenous packets will have the distribution of Y_j in future also. $\square$

Remark *If $s_T(i)$ for all i and s are lattice valued with the same lattice or at least one of them has a spread out distribution then the convergence in Theorem* 5 *above holds in total variation.*

From the above result we can easily see that the throughput of the ith TCP connection is

$$W_{max}(i) / \sum_{i=1}^{N} (W_{max}(i) E[s_T(i)] + E[Y_i]E[s])$$

packets/sec. where $E[Y_i] = \lambda E[s_T(i)]/(1 - \lambda E[s])$. The mean sojourn time of any TCP packet under stationarity equals

$$\sum_{i=1}^{N} W_{max}(i)(E[s_T(i)] + \lambda E[s]E[s_T(i)]/(1 - \lambda E[s]).$$

The mean sojourn time of the exogenous stream is also available in [8] and [36] but in this paper we will use the above approximation itself which turns out to be good enough in our simulations.

Appendix B
Convergence to the ODE: Single queue case

In this section we show the convergence claimed in (5). We will show it under the following assumptions. We assume that all packets are i.i.d. with an exponential distribution. The exogenous traffic is Poisson. The buffer can accommodate a finite number M of packets and the packets are lost only under RED control (no overflows). We will also assume $\Delta = 0$. We can generalize the proof to include the MMPP exogenous traffic, the phase type service times (different for the two classes) and exponential (and then phase type) propagation delays. Since any distribution (including deterministic) can be arbitrarily approximated by a phase type distribution and that for large buffers (and asymptotically exactly) the system with a buffer length B bits is close in performance to a system with buffer length $B/E[s]$ packets, a continuity argument (not provided here) should be able to provide the result in the full generality of section 2.

For the next theorem, we make one more approximation to the actual system. The probability of discarding the nth packet, if $\widehat{q}_n = \widehat{q}$ and $c_n = c$, is

$$p_n(\widehat{q}, c) \triangleq \frac{1}{\frac{T_{max}-T_{min}}{p_{max}(\widehat{q}-T_{min})} - c}$$

Thus $p_n(\widehat{q}, c)$ can be written as $p_n(\widehat{q}, 0)/(1 - cp_n(\widehat{q}, 0))$. In the following we will need $p_n(\widehat{q}, 0)$ to be continuously differentiable in $\widehat{q}$ for each c. This will happen if and only if $p_n(\widehat{q}, 0)$ is. However, $p_n(\widehat{q}, 0)$is continuously differentiable only when $\widehat{q} \neq T_{min}$ or T_{max} (see Fig. 8) below. But as the dotted curve shows, $p_n(\widehat{q}, 0)$ can be arbitrarily closely approximated by a function which is continuously differentiable at each $\widehat{q}$. Such a change will disturb the dynamics of the system only slightly (justified by Lemma 1 below). We make this modification for the next theorem.

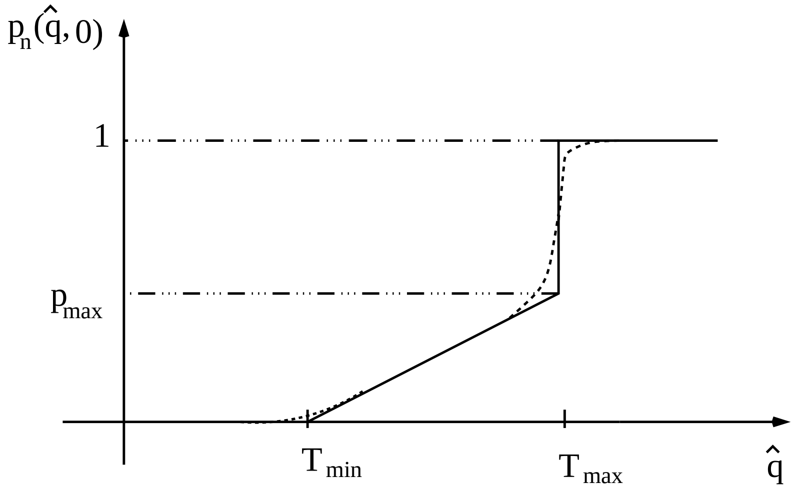

Figure 8:

To prove the convergence in (5) we will use the results from Kushner and Vazquez-Abad[26]. For this we first expand the state space of our process to make it Markovian. We consider the system at the arrival epochs (of both types of packets) to the queue. Now the system for a particular β will be denoted with a superscript β. Just before the nth arrival, we consider the system $(q_n^\beta, w_n^\beta, h_n^\beta, c_n^\beta, \widehat{q}_n^\beta, t_{n1}^\beta, ..., t_{nM}^\beta, t_n^\beta)$ where t_n^β is the type of the nth arrival (1=TCP and 2=exogenous) and t_{ni}^β is the type of the packet at position i in the queue ($t_{ni} = 0$ implies that the queue length is smaller than i). It is not difficult to verify that this expanded process is Markov (we also assume that if a TCP packet is discarded, its information reaches the TCP source immediately – this assumption is made only for convenience).

In the terminology of Kushner and Vazquez-Abad[26], we have $\epsilon = \beta$, $\theta_n^\epsilon = \hat{q}_n^\beta$, $Y_n^\epsilon = q_{n+1}^\beta - \hat{q}_n^\beta$, $\xi_n^\epsilon = (q_n^\beta, w_n^\beta, h_n^\beta, c_n^\beta, t_{n1}^\beta, ..., t_{nM}^\beta, t_n^\beta)$. We upper bound c_n^β by an arbitrarily large constant. In a practical system this will always happen because of finite word length. Then $\{\xi_n^\epsilon\}$ is a finite state Markov chain for any fixed $\hat{q}_n^\beta = \hat{q}$. It is also irreducible. Therefore, it has a unique stationary distribution which depends on $\hat{q}$. We will verify the following assumptions stated in [26] for convergence :

(3.2) $\{Y_n^\epsilon, n \geq 0, \epsilon \geq 0\}$ is uniformly integrable.

(3.4) $\{\xi_n^\epsilon, \theta_n^\epsilon, \epsilon \geq 0, n \geq 0, \}$ is tight.

(3.6) For each fixed θ, the transition matrix probabilities $P_n^\epsilon(\xi, .|\theta)$ converge weakly to that of a transition matrix $P(\xi, .|\theta)$ as $n \to \infty$, $\epsilon \to 0$, uniformly on compact sets (θ, ξ).

(3.7) $P(\xi, .|\theta)$ is weakly continuous in (ξ, θ).

(3.8) Let $\{\xi_n(\theta)\}$ be a Markov chain with transition matrix $P(\xi, .|\theta)$. Let $\mu(.|\theta)$ denote an invariant distributions for $\{\xi_n(\theta)\}$. Assume $\{\mu(.|\theta), \theta \in \Theta_1\}$ is tight for each compact set Θ_1.

(3.9) Let $G_n^\epsilon(\theta_n^\epsilon, \xi_n^\epsilon) = E[Y_n^\epsilon|\theta_n^\epsilon, \xi_n^\epsilon]$. Suppose there is a continuous function G such that for each $\delta > 0$, $\lim_\epsilon \lim_n supP[|G_n^n(\theta_n^\epsilon, \xi_n^\epsilon). - G(\theta_n^\epsilon, \xi_n^\epsilon)| \geq \delta] = 0$ and that for each compact set Θ there is a $k_0(\Theta) < \infty$ such that for all stationary processes $\{\xi_n(\theta)\}$

(3.10)

$$Sup_{\theta \in \Theta} E[|G(\theta, \xi_n(\theta)|] < k_0(\Theta) \tag{36}$$

(3.11a) For each θ, $\mu(.|\theta)$ is unique.

Theorem 6 *Under the above mentioned assumptions, the convergence in* (5) *holds.*

Proof. We verify the assumptions in [26] mentioned above for convergence to hold. Assumption (3.2) that $\{Y_n^\epsilon,\ n \geq 0,\ \epsilon > 0\}$ is uniformly integrable holds because $0 \leq \hat{q}_n^\beta \leq T_{max}$, $0 \leq q_n^\beta \leq M$. Similarly, because all other components of ξ_n^ϵ and θ_n^ϵ are also bounded, $\{\xi_n^\epsilon,\ \theta_n^\epsilon,\ n \geq 0,\ \epsilon > 0\}$ is tight. Markov chain $\{\xi_n^\epsilon\}$ for a fixed $\theta_n^\epsilon = \theta$ has a transition function independent of ϵ. This implies (3.6). Consider assumption (3.7). Except θ_n^ϵ, all other components of ξ_n^ϵ are discrete valued. Therefore, we need to show continuity of the transition function only with respect to $\theta_n^\epsilon = \hat{q}_n$. We prove this in Lemma 1. For assumption (3.8), we need to show that for any fixed $\hat{q}$, $\{\xi_n^\epsilon\}$ has a unique stationary distribution $\mu(\cdot|\theta)$. But this holds because $\{\xi_n^\epsilon\}$ is a finite state irreducible Markov chain. Also, from Lemma 1, we have the continuity of its transition matrix, with respect to $\hat{q}$, and from Lemma 2 below, we obtain the continuity of the corresponding stationary distributions. This implies that $\{\mu(\cdot|\theta),\ \theta \subset \Theta_1\}$ is a compact set for any compact set Θ_1. This of course provides the tightness of this set. The assumptions (3.9) and (3.10) hold by the finiteness of the state space of $\{\xi^{\epsilon}_n\}$. Assumption (3.11a) has been shown earlier. Thus all the assumptions for Theorem 3.1 in [26] are satisfied. $\square$

From [26] and using the results on ODE given below (see explanations and justifications at the end of section 2.3.1) we also obtain that π^{β} (the stationary distribution of $\widehat{q}^{\beta}_n$) converges weakly to δ_{q^*} as $\beta \downarrow 0$, where δ_{q^*} is a probability measure with all mass at q^*. Because of boundedness of $\widehat{q}^{\beta}_n$ this implies the convergence of all moments also.

Now we prove Lemma 1, needed in the proof of Theorem 5.

Lemma 1 Under the assumptions of this section the transition matrix of Markov chain $\{\xi^{\epsilon}_n, n \geq 0\}$ is a continuously differentiable function of $\widehat{q}$.

Proof. The proof involves considering each component of the transition matrix and showing its continuity as a function of $\widehat{q}$. We illustrate it by considering a few elements for the TCP Tahoe. For the other elements and also for TCP Reno one can show in the same way.

For a fixed $\widehat{q}$, consider the transition (with $m < M$)
$(q^{\beta}_n = m,\ w^{\beta}_n = w,\ h^{\beta}_n = n,\ c^{\beta}_n = c,\ t^{\beta}_{n1},\ \cdots,\ t^{\beta}_{nM},\ t^{\beta}_n) \rightarrow (m+1,\ w,\ n,\ c+1,\ t^{\beta}_{n+1i} = t^{\beta}_{n.i},$
$i \neq m+1,\ t^{\beta}_{n+1,m+1} = 1,\ t^{\beta}_{n+1} = 2)$.
This is the event that the $(n+1)$st arrival is from UDP and it is accepted in the queue. This has the probability $(1 - p(\widehat{q}, c))\lambda/(\lambda + \mu)$ where $p(\widehat{q}, c) = 1/(\bar{p}(\widehat{q}) - (c+1))$ and $\bar{p}(\widehat{q}) = (T_{max} - T_{min})/p_{max}(\widehat{q} - T_{min})$ for $\widehat{q} < T_{max}$ and probability zero if $\widehat{q} = T_{max}$. With the modification of the RED as mentioned above, this probability is a continuously differentiable function of $\widehat{q}$.

Consider the transition of the above mentioned state to $(m,\ w,\ n,\ c = 0,\ \cdots\ t^{\beta}_{n+1i} = t^{\beta}_{ni},$ $i = 1, \cdots, M,\ t^{\beta}_{n+1} = 2)$. This represents the transition due to the event that the $(n+1)$st arrival is from UDP and is not accepted in the queue. The probability for this transition is $\lambda/(\lambda + \mu)(\bar{p}(\widehat{q}) - (c+1))$ and is again a continuous function of $\widehat{q}$. The transition from the above state due to the next arrival being from TCP (this will happen if $t_{n1} = 1$ and its service gets completed) and the arriving packet is accepted is $(1 - p(\widehat{q}, c))\mu/(\mu + \lambda)$. This is also a continuously differentiable function of $\widehat{q}$. $\square$

Next we show the continuity and differentiability of π.

Lemma 2 Let $P(\theta)$ be a transition matrix for a finite state chain with N states. Let the elements of $P(\theta)$ be continuous functions of a finite dimensional vector θ. Let $\pi(\theta)$ be the unique stationary distribution of $P(\theta)$ for each θ in an open set Θ. If $P(\theta)$ is continuous in θ, then $\pi(\theta)$ is also continuous in θ. If $P(\theta)$ is continuously differentiable w.r.t. θ then $\pi(\theta)$ is also continuously differentiable.

Proof. We know that $\pi(\theta)$ satisfies $\pi(P(\theta) - I) = 0$ and $\sum \pi(i) = 1$. From these $N+1$ equations, we can form N linearly independent equations $\pi A(\theta) = x$ which provide the required $\pi(\theta)$. $A(\theta)$ is obtained from $(P(\theta) - I)$ by a linear transformation and is nonsingular and x is $(0, \cdots, 0, 1)$. Thus, elements of $A(\theta)$ are continuous (continuously differentiable) in θ if $P(\theta)$ is. Also, one can directly verify (e.g, by looking at the cofactors of $A(\theta)$), that if components of $A(\theta)$ are continuous (continuously differentiable) in θ then so are the components of $A^{-1}(\theta)x = \pi(\theta)$. $\square$

Finally we study the ODE (6). We have seen in Lemma 2 that the stationary distribution of $\{\xi_n^\epsilon\}$ is a continuously differentiable function of $\widehat{q}$. Therefore, the same holds for the stationary distribution of $\{q_n\}$. Since it is a bounded sequence, this implies that $E_\pi[q|z_t]$ is continuously differentiable as a function of z_t. Therefore, because z takes values in a bounded set, $E_\pi[q|z]$ is Lipschitz continuous in z. This provides a unique local solution of the ODE (6) for each initial condition $0 \leq a \leq T_{max}$. Next observe that $E_\pi[q|z_t] = 0$ for $z_t = T_{max} + \delta$. We will show below that $E_\pi[q|z_t]$ is a nonincreasing continuous function of z_t and $0 < E_\pi[q|z_t] < \infty$ for $z_t = 0$. Thus the equation $E_\pi[q|z] = z$ has a unique equilibrium point q^*. Also observe that when $z_t = \widehat{q} < q^*$ then $E[q|\widehat{q}] \geq E[q|q^*] = q^* > \widehat{q}$ and hence the solution z_t of the ODE strictly increases. By the upper boundedness of this trajectory (by q^*) it converges to an equilibrium point. But then, by uniqueness of the equilibrium point, $z_t \uparrow q^*$ as $t \to \infty$. Similarly when $z_t = \widehat{q} > q^*$ then $E[q|\widehat{q}] < \widehat{q}$ and z_t strictly decreases. Therefore, irrespective of the initial condition $z_0 = a \leq T_{max} + \delta$, z_t converges monotonically to q^*. This along with the existence and uniqueness of the local solution, implies that the solution extends to infinity, is unique and bounded.

Now we show that $E_\pi[q|z]$ is a nonincreasing continuous function of z. The continuity has already been shown. We show the nonincreasing part. We show it by assuming that c_n has been fixed to some value c. Then $p(z)$, the probability of discarding the nth packet is (for $T_{min} \leq z \leq T_{max}$)

$$p(z) = \frac{p_{max}(z - T_{min})}{(T_{max} - T_{min}) - c p_{max}(z - T_{min})}. \tag{37}$$

Thus $p(z)$ is obviously a nondecreasing function of z. Then for $\Delta = 0$,

$$E_\pi[q|z] = (1 + E[Y_1(z)])E[W(z)] \tag{38}$$

where $E[Y_1(z)] = \lambda(1 - p(z))E[s_T]/(1 - \lambda(1 - p(z))E[s])$. Since $E[Y_1(z)]$ is decreasing in $p(z)$, it is enough to show the same for $E[W(z)]$. When $\Delta > 0$, in (35) we replace $E[W(z)]$ by $E[q_T|z]$ where, from results provided in section 2.3.2,

$$E[q_T|z] = E[W|z] - \lambda_T \Delta$$

and $\lambda_T = (1 - \lambda E[s])/E(s_T]$. Thus, again it is sufficient to show that $E[W|z]$ is non increasing in z. But, this can be shown directly from the Padhye's formula provided in section 2.3.2.

Appendix C
ODE for the two queue case

Theorem 7 *The equilibrium points* (z_1^*, z_2^*) *of coupled ODE system* (20) *are local attractors for the system.*

Proof. One can show for the nonlinear ODE (20), (z_1^*, z_2^*) is a local attractor, if its linearization at (z_1^*, z_2^*) is an attractor.

Consider the linearization of the coupled ODE system (20) around the equilibrium point(s) (z_1^*, z_2^*)

$$\begin{aligned}\dot{\tilde{z}}_1 &= \frac{\partial}{\partial z_1}\{E[q(1)|z_1, z_2] - z_1\}\ \tilde{z}_1 + \frac{\partial}{\partial z_2}\{E[q(1)|z_1, z_2] - z_1\}\ \tilde{z}_2 \\ \dot{\tilde{z}}_2 &= \frac{\partial}{\partial z_1}\{E[q(2)|z_1, z_2] - z_2\}\ \tilde{z}_1 + \frac{\partial}{\partial z_2}\{E[q(2)|z_1, z_2] - z_2\}\ \tilde{z}_2 \qquad (39)\end{aligned}$$

where $\tilde{z}_1 = z_1 - z_1^*$ and $\tilde{z}_2 = z_2 - z_2^*$,the partial derivatives being evaluated at (z_1^*, z_2^*). This system of linear equations is of the form $\dot{\tilde{z}} = A\tilde{z}$ where $\tilde{z} = [\tilde{z}_1, \tilde{z}_2]^T$, whose dynamics is well studied. We use the results given in [32]. Accordingly, let $\delta = \det A$, $\tau = \mathrm{tr} A$. Then from [32] if $\delta > 0$ we have a stable node or focus at the origin if $\tau < 0$. If $\delta < 0$ we have a saddle point at the origin. If $\delta > 0$ and $\tau = 0$ we have a center at the origin. In the following we apply these results to our system.

We consider the two cases: symmetric, $\rho_1 = \rho_2$, and asymmetric, $\rho_1 > \rho_2$ (the other case $\rho_1 < \rho_2$ can be similarly taken care of). The case $\rho_1 = \rho_2$ can be subdivided into the subcases:

(1) $\frac{W_{max} - \frac{(1-\rho_1)}{E[s_T]}\Delta}{2(1-\rho_1)} < T_{min}(1)$, $\frac{W_{max} - \frac{(1-\rho_1)}{E[s_T]}\Delta}{2(1-\rho_1)} < T_{min}(2)$;

(2) $\frac{W_{max} - \frac{(1-\rho_1)}{E[s_T]}\Delta}{2(1-\rho_1)} \geq T_{min}(1)$, $\frac{W_{max} - \frac{(1-\rho_1)}{E[s_T]}\Delta}{2(1-\rho_1)} \geq T_{min}(2)$;

(3) $T_{min}(1) < \frac{W_{max} - \frac{(1-\rho_1)}{E[s_T]}\Delta}{2(1-\rho_1)} < T_{min}(2)$. (case symmetric to this can be similarly taken care of)

We illustrate for subcase (2) how the stability of the linearized equations could be studied. For the other cases the methods used are similar.

In subcase (2), the equilibrium point (z_1^*, z_2^*) is such that $T_{min}(1) \leq z_1^* \leq T_{max}(1)$ and $T_{min}(2) \leq z_2^* \leq T_{max}(2)$ where (z_1^*, z_2^*) is determined by solving the equations

$$\begin{aligned}z_1^* = \frac{E[W|z_1^*, z_2^*]}{2(1-\rho_1(1-p_1))} - \frac{\Delta}{2E[s_T]} &= E_\pi[q(1)|z_1^*, z_2^*], \\ z_2^* = \frac{E[W|z_1^*, z_2^*]}{2(1-\rho_2(1-p_2))} - \frac{\Delta}{2E[s_T]} &= E_\pi[q(2)|z_1^*, z_2^*] \qquad (40)\end{aligned}$$

where $p_i = \frac{p_{max}(i)[z_i^* - T_{min}(i)]}{T_{max}(i) - T_{min}(i)}$, $i = 1, 2$. The following are the linearized equations

$$\begin{aligned}\dot{\tilde{z}}_1 &= (-\alpha_{11} - 1)\ \tilde{z}_1 + (-\alpha_{12})\ \tilde{z}_2, \\ \dot{\tilde{z}}_2 &= (-\alpha_{21})\ \tilde{z}_1 + (-\alpha_{22} - 1)\ \tilde{z}_2 \qquad (41)\end{aligned}$$

where $\alpha_{ij} \triangleq -\frac{\partial}{\partial z_j} E[q(i)|z_1, z_2], i = 1, 2, j = 1, 2$, are strictly positive. Thus $\delta = (\alpha_{11} + 1)(\alpha_{22} + 1) - \alpha_{12}\alpha_{21}$ and $\tau = -\alpha_{11} - \alpha_{22} - 2 < 0$. For this case explicit computations show $\alpha_{11}\alpha_{22} = \alpha_{12}\alpha_{21}$, so that $\delta = 1 + \alpha_{11} + \alpha_{22} > 0$. Hence, we always have a stable node or a stable focus in this case.

Similarly we can carry out the analysis for subcases (1) and (3) and we find that in both the subcases we have stability. Thus for the case $\rho_1 = \rho_2$, we find that the equilibrium point is always a local attractor.

The case $\rho_1 > \rho_2$ can be similarly subdivided and its stability studied. We note here that we need to use the approximations used in Section 3.2. For all the subcases we have checked that the equilibrium point is a local attractor. □

Thus for all the cases we find that the equilibrium points (z_1^*, z_2^*) are local attractors.

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Table 1: $\beta = 10^{-3}$, Throughput (pkts/sec), Sojourn time (secs.), $W_{max} = 30$, $\Delta = 5$ms

	TCP t'put		Exog. t'put	
	Theory	Simul.	Theory	Simul.
$\hat{z}^{\beta}(t) \leq T_{min}$	833.33	918	833.33	751.00
$\hat{z}^{\beta}(t) = 8.458$ stationary value	862.17	812	804.51	832.6

	TCP soj. time		Exog. soj. time	
	Theory	Simul.	Theory	Simul.
$\hat{z}^{\beta}(t) \leq T_{min}$	0.031	0.027	0.031	0.027
$\hat{z}^{\beta}(t) = 8.458$ stationary value	0.0062	0.0059	0.0062	0.0061

Table 2: $\beta = 10^{-4}$, Throughput (pkts/sec), Sojourn time (secs.), $W_{max} = 30$, $\Delta = 5$ms

	TCP t'put		Exog. t'put	
	Theory	Simul.	Theory	Simul.
$\hat{z}^{\beta}(t) \leq T_{min}$	833.33	855.00	833.33	811.67
$\hat{z}^{\beta}(t) = 8.458$ stationary value	862.17	816.00	804.51	834.84

	TCP soj. time		Exog. soj. time	
	Theory	Simul.	Theory	Simul.
$\hat{z}^{\beta}(t) \leq T_{min}$	0.031	0.028	0.031	0.029
$\hat{z}^{\beta}(t) = 8.458$ stationary value	0.0062	0.0059	0.0062	0.0060

Table 3: $\beta = 10^{-4}$, MMPP case, Throughput (pkts/sec), Sojourn time (secs.), $W_{max} = 30$, $\Delta = 5$ms

	TCP t'put		Exog. t'put	
	Theory	Simul.	Theory	Simul.
$\hat{z}^{\beta}(t) \leq T_{min}$	833.33	780	833.33	809.76
$\hat{z}^{\beta}(t) = 8.458$ stationary value	862.17	688	818	840

	TCP soj. time		Exog. soj. time	
	Theory	Simul.	Theory	Simul.
$\hat{z}^{\beta}(t) \leq T_{min}$	0.031	0.030	0.031	0.032
$\hat{z}^{\beta}(t) = 8.458$ stationary value	0.0062	0.0064	0.0062	0.0078

Table 4: $\beta = 10^{-4}$, Throughput (pkts/sec), Sojourn time (secs.), $W_{max} = 50$, $\Delta = 10$ms, nonbottleneck case.

	TCP t'put		Exo.1 t'put		TCP soj. time Q_1		Exo. soj. time Q_1	
	The.	Sim.	The.	Sim.	The.	Sim.	The.	Sim.
$\hat{z}_1^{\beta}(t) \leq T_{min}$	833.3	840	833.3	817.5	0.025	0.028	0.025	0.028
$\hat{z}_1^{\beta}(t) = 5.668$ stationary value	844.2	735.4	822.5	844	0.004	0.005	0.004	0.005

	Exo.2 t'put		TCP soj. time Q_2		Exo. soj. time Q_2	
	The.	Sim.	The.	Sim.	The.	Sim.
$\hat{z}^{\beta}(t) \leq T_{min}$	833.3	830	0.025	0.021	0.025	0.021
$\hat{z}^{\beta}(t) = 5.668$ stationary value	822.5	844	0.004	0.005	0.004	0.005

Table 5: $\beta = 10^{-4}$, Throughput in pkts/sec, Sojourn time in secs., $W_{max} = 30$, $\Delta = 10$ms, bottleneck case

	TCP t'put		Exo.1 t'put		TCP soj. time Q_1		Exo. soj. time Q_1	
	The.	Sim.	The.	Sim.	The.	Sim.	The.	Sim.
$\hat{z}_1^{\beta}(t) \leq T_{min}$	833.3	815	833.3	855	0.025	0.026	0.025	0.026
$\hat{z}_1^{\beta}(t) = 6.605$ stationary value	846.7	781.3	820	821.7	0.005	0.005	0.005	0.005

Exo.2 t'put		Exo. soj. time Q_2		TCP soj. time Q_2	
The.	Sim.	The.	Sim.	The.	Sim.
666.7	666.6	0.001	0.002	0.001	0.002

Table 6: $\beta = 10^{-4}$, Throughput in pkts/sec, Sojourn time in secs., $\rho_1 = 0.6$, $\rho_2 = 0.3$

	TCP1 t'put		TCP2 t'put		Exo.1 t'put	
	The.	Sim.	The.	Sim.	The.	Sim.
$\hat{z}_1^{\beta}(t) \leq T_{min}(1)$	323	326.7	343.5	351.7	1000	998.3
$\hat{z}_1^{\beta}(t) = 11.5$ stat. val.	247	237.8	419.8	371.7	993	996.2

	TCP1 soj. time Q_1		TCP2 soj. time		Exo.1 soj. time	
	The.	Sim.	The.	Sim.	The.	Sim.
$\hat{z}_1^{\beta}(t) \leq T_{min}(1)$	0.047	0.046	0.047	0.046	0.047	0.046
$\hat{z}_1^{\beta}(t) = 11.5$ stat. val.	0.008	0.009	0.008	0.009	0.008	0.008

	TCP3 t'put		TCP4 t'put		Exo.2 t'put	
	The.	Sim.	The.	Sim.	The.	Sim.
$\hat{z}_2^{\beta}(t) \leq T_{min}(2)$	451.3	450	392.3	397	500	493
$\hat{z}_2^{\beta}(t) = 22.4$ stat. val.	507.3	500.7	412.5	426.1	497	494.3

	TCP3 soj. time		TCP4 soj. time		Exo.2 soj. time		TCP1 soj. time Q_2	
	The.	Sim.	The.	Sim.	The.	Sim.	The.	Sim.
$\hat{z}_2^{\beta}(t) \leq T_{min}(2)$	0.037	0.037	0.037	0.037	0.037	0.037	0.037	0.037
$\hat{z}_2^{\beta}(t) = 22.4$ stat. val.	0.013	0.014	0.013	0.014	0.013	0.014	0.013	0.014

Table 7: $\beta = 10^{-4}$, Throughput in pkts/sec, Sojourn time in secs., $\rho_1 = 0.3$, $\rho_2 = 0.6$

	TCP 1 t'put		TCP 3 t'put		TCP4 t'put		Exo. 2 t'put	
	The.	Sim.	The.	Sim.	The.	Sim.	The.	Sim.
$\hat{z}_2^{\beta}(t) \leq T_{min}(2)$	261.3	263.9	209.7	230.6	203.5	183.3	1000	986.1
$\hat{z}_2^{\beta}(t) = 28.4$ stationary value	208	207	250.7	236.6	208	218	979	991.9

	TCP1 soj. time Q_2		TCP3 soj. time		TCP4 soj. time		Exo.2 soj. time	
	The.	Sim.	The.	Sim.	The.	Sim.	The.	Sim.
$\hat{z}_2^{\beta}(t) \leq T_{min}(2)$	0.113	0.107	0.113	0.106	0.113	0.106	0.113	0.106
$\hat{z}_2^{\beta}(t) = 28.4$ stationary value	0.019	0.017	0.019	0.017	0.019	0.017	0.019	0.017

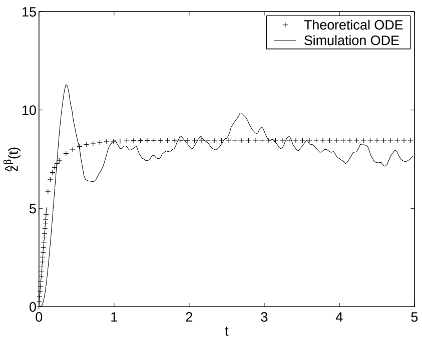

Figure 9: $\hat{z}^{\beta}(t)$,$\beta = 10^{-3}$, $W_{max} = 30$, $\Delta = 5$ms

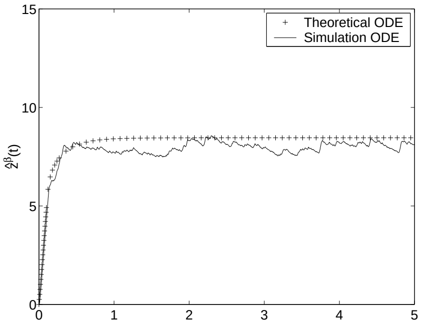

Figure 10: $\hat{z}^{\beta}(t)$,$\beta = 10^{-4}$, $W_{max} = 30$, $\Delta = 5$ms

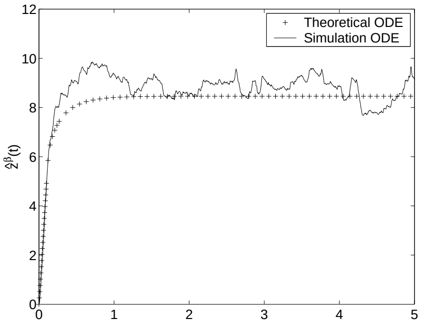

Figure 11: $\hat{z}^{\beta}(t)$,$\beta = 10^{-4}$, MMPP case, $W_{max} = 30$, $\Delta = 5$ms

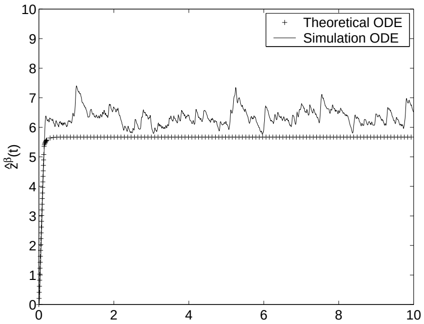

Figure 12: $\hat{z}^{\beta}(t)$,$\beta = 10^{-4}$, Q_1, nonbottleneck case, $W_{max} = 50$, $\Delta = 10$ms

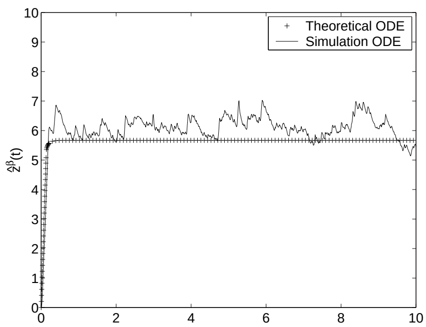

Figure 13: $\hat{z}^{\beta}(t)$,$\beta = 10^{-4}$, Q_2, nonbottleneck case, $W_{max} = 50$, $\Delta = 10$ms

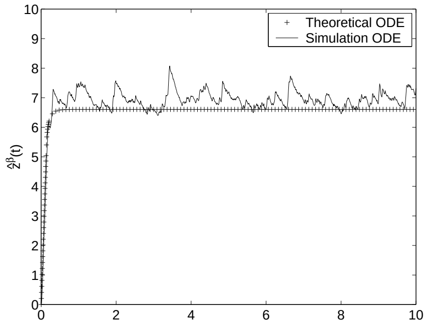

Figure 14: $\hat{z}^{\beta}(t)$,$\beta = 10^{-4}$, Q_1, bottleneck case, $W_{max} = 30$, $\Delta = 10$ms

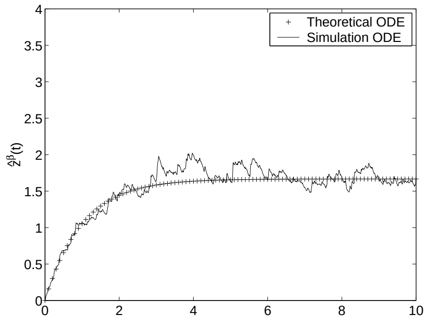

Figure 15: $\hat{z}^{\beta}(t)$,$\beta = 10^{-4}$, Q_2, bottleneck case, $W_{max} = 30$, $\Delta = 10$ms

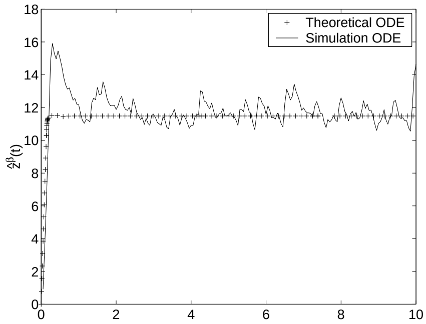

Figure 16: $\hat{z}^{\beta}(t)$,$\beta = 10^{-4}$, Q_1, $\rho_1 = 0.6$, $\rho_2 = 0.3$, Multiple TCP, Multiple queue case

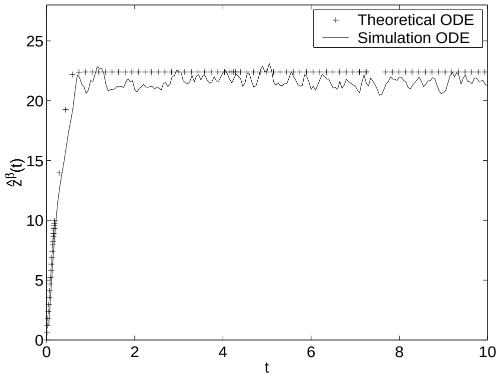

Figure 17: $\hat{z}^{\beta}(t)$,$\beta = 10^{-4}$, Q_2, $\rho_1 = 0.6$, $\rho_2 = 0.3$, Multiple TCP, Multiple queue case

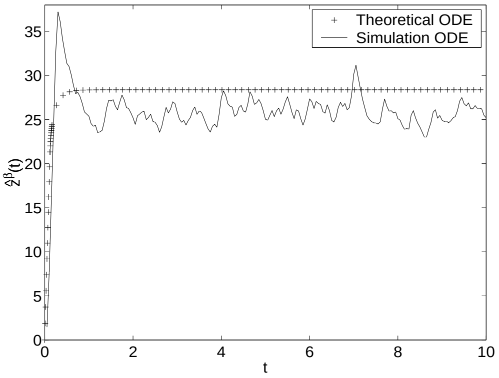

Figure 18: $\hat{z}^{\beta}(t)$,$\beta = 10^{-4}$, Q_2, $\rho_1 = 0.3$, $\rho_2 = 0.6$, Multiple TCP, Multiple queue case