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Performance analysis of a slotted-ALOHA protocol on a capture channel with fading

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We consider the slotted ALOHA protocol on a channel with a capture effect. There are $M < \infty$ users each with an infinite buffer. If in a slot, i packets are transmitted, then the probability of a successful reception of a packet is q_i. This model contains the CDMA protocols as special cases. We obtain sufficient rate conditions, which are close to necessary for stability of the system, when the arrival streams are stationary ergodic. Under the same rate conditions, for general regenerative arrival streams, we obtain the rates of convergence to stationarity, finiteness of stationary moments and various functional limit theorems. Our arrival streams contain all the traffic models suggested in the recent literature, including the ones which display long range dependence. We also obtain bounds on the stationary moments of waiting times which can be tight under realistic conditions. Finally, we obtain several results on the transient performance of the system, e.g., first time to overflow and the limits of the overflow process. We also extend the above results to the case of a capture channel exhibiting Markov modulated fading. Most of our results and proofs will be shown to hold also for the slotted ALOHA protocol without capture.

Keywords: multiple access, CDMA, rates of convergence, stability, functional limit theorems, transient analysis, Markov-modulated capture channel

1. Introduction

The problem considered in this paper is motivated by the following situation in mobile cellular communication networks. Mobile units or nodes, spread out over a given region, transmit packets to the base station in that region in a slotted manner. If i nodes transmit packets in a slot, one or more of these could be received error-free by the base station. The multiple successful receptions are made possible by the powerful codes used for their transmissions. In particular, these could be the orthogonal codes used in CDMA [20]. The packets which are not received successfully by the base station have to be retransmitted in a later slot.

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The above protocol, called *slotted-ALOHA with capture effect*, has been extensively studied in the literature. For an analysis of this protocol operating in mobile radio environment, see [16,40]. In infinite node case, the stability of this system has been proved and some retransmission algorithms to improve the stability region have been proposed in [12,13,40].

In this paper, we consider $M < \infty$ number of nodes. Node i receives packets at the rate of λ_i packets per slot and has an infinite buffer to store these packets. Arrival processes to different nodes are independent of each other, but the arrival stream to a particular node can be general stationary, ergodic. If any node has packets to transmit at the beginning of a slot, it transmits the head of the line packet in that slot. The channel is characterised as follows. If i nodes transmit in a slot, the probability of successful reception (called *capture probability*) of a packet is q_i. The successes in a slot are independent of each other and the q_i's satisfy $q_1 \geqslant q_2 \geqslant \cdots \geqslant q_M > 0$. Our study can be used for various implementations of CDMA systems, as long as we can finally obtain the capture probabilities q_i. Later on we will also include the effect of fading in our analysis. Although the slotted ALOHA (without capture) is not included in the above model we will comment on the results and most of the proofs which go through for this protocol also.

Given a set $\{q_i,\ 1 \leqslant i \leqslant M\}$, we aim to provide necessary and sufficient conditions for stability of the system. For such interacting queues these conditions are not available in the literature. The conditions we obtain provide upper bounds for stability on the arrival rates λ_i of the users. However, against as in most of the queueing literature, the upper bounds obtained are functions of the *distributions* of the arrival streams. Furthermore, these can be obtained, in general, only via estimation through simulation. Therefore, we then provide sufficient conditions, which can be close to necessary, and are explicitly calculable from the arrival rates only. Next we obtain rates of convergence to stationary distributions when the arrival streams are regenerative. Such generality on the assumptions of the arrival streams (as against *i.i.d.* assumptions) is important in high speed networks, where a realistic arrival stream can be Markov modulated or could have long range dependence. Several models of long range dependence described in the literature have regenerative structure, e.g., the ON–OFF model in Likhanov and Tsybakov [21]. Under the same generality, we obtain conditions for finiteness of stationary moments of waiting times and various other limit theorems (e.g., central limit theorem). We also obtain bounds on the moments of stationary delays which can be tight under some realistic conditions for the CDMA systems. Next we obtain several results on transient performance measures, e.g., first time to overflow, the limiting distributions of the process of overflow epochs, and time to empty a queue. These performance measures are increasingly being studied in some queueing systems (although for a complicated system of interacting queues, this seems to be the first study) because of their importance in packetized voice and video. Finally, we will extend our results to the case where the channel exhibits a Markov modulated fading. All our results are new even in the special case of a slotted ALOHA or a CDMA system.

The problem with a capture channel has received considerable attention in the literature so far [12,13,16,22,39,40]. The expressions for the capture probabilities, their insensitivity to the underlying channel for a large class of channels and their limits as the number of users goes to infinity are some of the issues considered in [16,40]. Stability conditions for the case of infinite nodes are obtained in [12,40]. In [22] the throughput for a finite number of nodes is obtained, although the authors assume that a backlogged node does not generate any packet. This is not a realistic assumption. In short, a satisfactory stability and delay analysis for the system we consider does not appear in the past literature. We attempt to fill the gap in this paper. Our model of a finite number of users each with an infinite buffer can be a more realisic model in many practical situations. Also our techniques are significantly different from those used in the infinite node case.

Our main motivation for the present problem is to analyse a CDMA system, which is being proposed as an alternative to the TDMA protocol in wireless networks [6,7,14,20,25]. In a CDMA system, transmissions to a base station by different nodes are modulated by different spreading sequences, which are nearly orthogonal to each other. Each transmission can then be captured by the base station receiver after applying the same spreading sequence. We will assume that the receivers at the base station detect (estimate) the codes for each user individually (this is the common scheme although joint decoding for different users is being considered in some recent studies). In practice a perfect capture is not possible since the spreading sequences are not strictly orthogonal amongst themselves, and the noise, usually modelled as additive white Gaussian noise, causes transmission errors. We assume that if i packets are transmitted in a slot, the capture probability is q_i. Now we describe the calculation of q_i for a typical CDMA scheme. For a total of $i-1$ interferers, the probability of bit error $P_{\mathrm{e}}(i) \approx Q(\sqrt{\mathrm{SNR}})$, where

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} \mathrm{e}^{-u^2/2}\, \mathrm{d}u \quad \text{and} \quad \mathrm{SNR} = \frac{3P_0 S}{(i-1)P_0 + I_{\text{other}}}.$$

In this formula S is the spread factor, P_0 is the received power and I_{other} is the total interference from outside the cell. The significance of these quantities and their typical values can be found in [3,38]. Now q_i can be computed as

$$q_i = \sum_{j=0}^{t} \binom{L}{j} \bigl(P_{\mathrm{e}}(i)\bigr)^j \bigl(1 - P_{\mathrm{e}}(i)\bigr)^{L-j},$$

for an L bit packet with upto t correctable errors (this is achieved using forward error correction coding), assuming uncorrelated bit errors. For a typical system, q_i is close to 1 upto a certain point and then drops sharply to zero. In this paper, we assume the knowledge of q_i's. In our next paper [26], we compare a CDMA system with a TDMA system based on their stability regions and the delay bounds. That comparison uses some key results from this paper.

The channel model considered so far assumes that there is no time-varying attenuation introduced by the channel. However, in mobile wireless communications, the channels undergo multipath fading resulting in rapidly changing attenuation of the received signal. The attenuation is due to the fact that many versions of the transmitted signal superimpose on each other resulting in constructive or destructive interference. Also, it has been found that, if the packet length is small, then several successively transmitted packets undergo correlated attenuation. To model this dependence, Markov models have been proposed in the literature [19,36]. We extend our analysis to this model. The channel is assumed to be in any one of the L states $\{1, 2, \ldots, L\}$. The state of the channel during slot k is given by δ_k, where $\{\delta_k\}$ is an irreducible, aperiodic discrete-time Markov chain with the state space $\{1, 2, \ldots, L\}$. If j packets are transmitted in a slot, and the channel state is l during that slot, a packet is successfully received with probability $q_j^{(l)}$ independently of anything else. Note that if $\delta_k = l$, then the attenuation during the slot is at a level, say, α_l. For this level, packet error probability can be found as described in [26]. This model corresponds to the situation in which all the nodes undergo exactly the same amount of attenuation. We also model the situation in which all the nodes undergo independent and identically distributed fading. For this purpose, we have M different Markov chains, $\{(\delta_k(i)),\ 1 \leqslant i \leqslant M\}$, which are mutually independent and identically distributed, with $\{\delta_k(i)\}$ determining the channel state during slot k for user i. Now the packet success probability of any user depends on how many other users transmit and its own channel state.

The rest of the paper is planned as follows. Section 2 provides the stability condition. Rates of convergence for regenerarive input traffic are obtained in section 3. In this section, we also provide the finiteness of stationary moments of the queue length process and various other limit theorems, e.g., CLT. Section 4 extends all the previous results to the delay process, which is of more practical concern to a user. It also includes some bounds. Section 5 contains transient analysis. Section 6 generalizes several of these results to a Markov modulated channel.

2. The model and its stability

Let there be $M < \infty$ nodes transmitting on a common channel. Each node has an infinite buffer to store the incoming packets. All the packets are of fixed size and the time-axis is divided into fixed length slots such that exactly one packet can be transmitted in a slot. If the packet sizes are random and form jointly stationary distributions, all the results will go through (when we require $\{x_k(i)\}$, defined below, to be *i.i.d.*, then the packet sizes should also be *i.i.d.*) with obvious changes. If in a slot, i nodes transmit on the channel, any of those transmissions is successful with probability q_i independently of anything else. This holds for a CDMA system if at the base station for each user there is a separate decoding. This is true for traditional CDMA systems although more recently receivers decoding several users' codes jointly have also been studied. The unsuccessful packets are retransmitted in the next slot. We make the following natural assumption:

$$q_1 \geqslant q_2 \geqslant \cdots \geqslant q_M > 0. \tag{2.1}$$

We will use the following notation (all random variables are defined on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$):

- $x_k(i) =$ number of new packets received in the kth slot at the ith node.
- $\lambda_i = \mathbb{E}[x(i)]$, assuming $\{x_k(i)\}$ to be stationary.
- $$T_k(i) = \begin{cases} 1 & \text{if the packet transmission is successful during the } k\text{th slot} \\ & \text{at the } i\text{th node, provided there was a transmission,} \\ 0 & \text{otherwise.} \end{cases}$$
- $z_k(i) =$ queue length at the beginning of the kth slot at the ith node.
- x_k, T_k, z_k are M-dimensional vectors with $x_k(i)$ being the ith component of x_k (similarly for others).
- $\mathcal{L}$ denotes the process $\{z_k, \ -\infty < k < \infty\}$.
- $\ell(z_k) =$ cardinality$\{z_k(j) > 0\colon\ 1 \leqslant j \leqslant M\}$.

We will assume the sequences $\{(x_k(i))\}$, $1 \leqslant i \leqslant M$, to be independent of each other, although this condition is not required for theorem 2.1. Since our system can be described by the equation

$$z_{k+1}(j) = \bigl(z_k(j) - T_k(j)\bigr)^+ + x_k(j), \tag{2.2}$$

the above assumptions imply the following. For $1 \leqslant i \leqslant M$, $1 \leqslant j \leqslant M$,

$$\begin{aligned} \mathbb{P}\bigl(T_k(j) = 1 \mid l(z_k) = i\bigr) &= q_i, \\ \mathbb{P}\bigl(T_k(j) = 0 \mid l(z_k) = i\bigr) &= 1 - q_i. \end{aligned}$$

Also, under our assumptions, given $l(z_k)$, $\{T_k(j),\ 1 \leqslant j \leqslant M\}$ are independent and T_k is independent of $T_{k-1}, T_{k-2}, \ldots, T_1$.

Henceforth assume, $\{x_k\}$ is strictly stationary. In the following proof we use the Loynes' approach [23]. For slotted ALOHA case (without capture), this approach for stability was first used in [27].

Theorem 2.1. Suppose $\{x_k\}$ is stationary. Then there exists a stationary solution $\{z_k\}$ (with distribution π) and starting from $z_0 \equiv 0$, z_k converges weakly to it. If $\{x_k\}$ is also ergodic, and ($T(i)$ is a generic *r.v.* of sequence $\{T_k(i),\ k \geqslant 1\}$)

$$\lambda_i < \mathbb{E}_\pi\bigl[T(i)\bigr], \quad 1 \leqslant i \leqslant M, \tag{2.3}$$

then the distribution π of this stationary solution is proper. (Observe that since $T(i)$ are $0, 1$ valued, $0 \leqslant \mathbb{E}_\pi[T(i)] \leqslant 1$, even if π is not a proper distribution.)

Proof. To prove the result we will use the following construction. Let $\{U_k(i), k \geqslant 0\}, 1 \leqslant i \leqslant M$, be *i.i.d.* sequences, each independent of another and $U_k(i)$ has uniform distribution on $[0,1]$ for all i. If $\ell(z_k) = j_0$, then define $T_k(i) = \mathbf{1}\{U_k(i) \leqslant q_{j_0}\}$. Thus, $T_k(i)$ is a nonincreasing function of z_k. Also, now the $\{z_k\}$ sequence obtained from (2.2) can be rewritten as

$$z_{k+1} = g\bigl(z_k, (U_k, x_k)\bigr),$$

where g is monotonically increasing in the first component and $\{(U_k, x_k)\}$ is a stationary sequence. Thus, by Loynes' lemma [23], this equation has a stationary solution to which z_k converges weakly if $z_0 \equiv 0$. In fact again by monotonicity, this is the (stochastically) smallest stationary distribution $\{z_k\}$ has.

To prove the next part of the theorem, again consider equation (2.2) with $\{(z_k, T_k, x_k)\}$ as its stationary version obtained above. When $\{x_k\}$ is ergodic, this stationary version of $\{(z_k, T_k, x_k)\}$ is also ergodic. Considering each queue j separately, this is just the equation of a $\mathrm{G}/\mathrm{G}/1$ queue (written in a slightly modified form). Hence, under (2.3), $\{z_k(i), T_k(i)\}$ has a proper unique distribution for each i. Then the limiting stationary measure of $\{(z_k, T_k, x_k)\}$ obtained above is also a proper distribution. The necessity of the rate condition also follows from the necessity in a $\mathrm{G}/\mathrm{G}/1$ queue. $\square$

In the sequel, we call a system *stable*, if the queue length process of that system posseses a proper stationary distribution.

For many practical systems, it is more appropriate to model the arrival process as a periodic input process, i.e., the finite dimensional distributions of $\{x_n, x_{n+1}, \ldots, x_{n+m}\}$ and $\{x_{n+d}, x_{n+d+1}, \ldots, x_{n+m+d}\}$ are the same for some integer $d \geqslant 1$ and for all n and m. In this case, it can be shown that, z_k has periodic stationary distributions with period d and the above theorem is valid with λ_i and $\mathbb{E}_\pi[T(i)]$ replaced by their corresponding averages under these d periodic stationary distributions.

If (2.3) is satisfied for some i and $\{x_k\}$ are ergodic, then that particular queue will be stable, irrespective of whether the whole system is stable or not.

Though, condition (2.3) is difficult to verify directly, it does give us the following sufficient condition. This follows from the inequality $\mathbb{E}_\pi[T_i] \geqslant q_M$. Furthermore, theorem 2.1 will be needed in the proofs of theorems 2.3 and 2.4. The conditions in theorem 2.3 are somewhat more explicit. The most easily verifiable conditions will be provided in theorem 2.5.

Corollary 2.2. System $\mathcal{L}$ is stable if

$$\lambda_i < q_M, \quad 1 \leqslant i \leqslant M. \tag{2.4}$$

One of the objectives of this paper is to improve the sufficient conditions in (2.4) and make them close to necessary. This is achieved in theorems 2.3 and 2.5 below. Without loss of generality, for the rest of the paper assume

$$0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_M. \tag{2.5}$$

For $1 \leqslant i \leqslant M$, let $\mathcal{L}^{(i)}$ denote the queue length process

$$z = \left\{ \left(z_k(1), z_k(2), \ldots, z_k(M) \right), \ -\infty < k < \infty \right\}$$

under the assumption that $z_k(i+1) = z_k(i+2) = \cdots = z_k(M) = \infty$ for all k. Otherwise, the system evolves as before. Now the stability of $\mathcal{L}^{(i)}$ refers to the stability of the first i queues only, that is, only the first i queues need to have a proper stationary distribution. Then $\mathcal{L}^{(M)}$ is the original system $\mathcal{L}$.

Note that each $\mathcal{L}^{(i)}$ posseses a stationary distribution $\pi^{(i)}$ by theorem 2.1. This may or may not be proper. Define for $j = 0, 1, \ldots, i$ and $i = 1, 2, \ldots, M-1$,

$$\begin{aligned} \rho_j^{(i)} &= \mathbb{P}_{\pi^{(i)}} \quad (\text{exactly } j \text{ queues are empty among the first } i \text{ queues}), \\ r_1 &= q_M, \qquad r_{i+1} = \mathbb{E}_{\pi^{(i)}}\left[T(j)\right], \quad 1 \leqslant i \leqslant M-1. \end{aligned}$$

Thus, r_{i+1} is the average service rate (per slot) under stationarity of $\mathcal{L}^{(i)}$. It is always well-defined, whether $\pi^{(i)}$ is proper or not. Also, observe that under stationarity of $\mathcal{L}^{(i)}$ the server operates at rate q_{M-l} with probability $\rho_l^{(i)}$. Therefore,

$$r_{i+1} = \sum_{l=0}^{i} \rho_l^{(i)} q_{M-l}, \quad i = 1, 2, \ldots, M-1, \tag{2.6}$$

where r_2 can be explicitly computed as

$$\rho_0^{(1)} = \frac{\lambda_1}{q_M}, \qquad \rho_1^{(1)} = 1 - \rho_0^{(1)}, \qquad r_2 = \frac{\lambda_1}{q_M} q_M + \left(1 - \frac{\lambda_1}{q_M}\right) q_{M-1}.$$

We now give a result by which the stability of each $\mathcal{L}^{(i)}$ can be checked step by step, leading to the stability of $\mathcal{L}$ itself.

Theorem 2.3. If

$$\lambda_j < r_j, \quad 1 \leqslant j \leqslant i,$$

then $\mathcal{L}^{(i)}$ is stable. In particular, $\mathcal{L}$ is stable if

$$\lambda_j < r_j, \quad 1 \leqslant j \leqslant M.$$

Proof. From corrollary 2.2 we know that $\mathcal{L}^{(1)}$ is stable if $\lambda_1 < r_1 = q_M$. Now we prove that the stability of $\mathcal{L}^{(i-1)}$ and $\lambda_i < r_i$ imply the stability of $\mathcal{L}^{(i)}$.

First observe that by theorem 2.1, $\mathcal{L}^{(j)}$, $1 \leqslant j \leqslant M$, has a stationary distribution $\pi^{(j)}$. We only need to prove the properness of this distribution. We also know that if in system $\mathcal{L}^{(j)}$, the first j queues are empty at $k = 0$, then the system queues converge (stochastically monotonically) weakly to $\pi^{(j)}$. Now start both $\mathcal{L}^{(i)}$ and $\mathcal{L}^{(i-1)}$ with the first i (respectively $i-1$) queues empty and construct (as in theorem 2.1) the sequences $\{T_k(j)\}$ for both the systems from the same sequences $\{U_k(j)\}$ and $\{x_k\}$ (denoted by $\{T_k^{(i)}(j)\}$ and $\{T_k^{(i-1)}(j)\}$, respectively). Then the first $i-1$ queues in $\mathcal{L}^{(i)}$ are a.s. upper-bounded by the first $i-1$ queues in $\mathcal{L}^{(i-1)}$. Since by assumption the $i-1$ queues in $\mathcal{L}^{(i-1)}$ converge weakly to the proper stationary distribution $\pi^{(i-1)}$ (in fact all finite dim distributions converge), the limiting stationary distribution of the first $i-1$ queues in $\mathcal{L}^{(i)}$ is also proper. We only need to show the properness of the limiting distribution of queue i in $\mathcal{L}^{(i)}$. But the $\{T_k^{(i)}(i)\}$ are a.s. lower bounded by sequence $\mathbf{1}\{U_k(i) \leqslant q_{\ell(z_k^{(i-1)})}\}$, which is converging stochastically monotonically weakly to $\{T_k^{(i-1)}(j)\}$ under the stationary distribution $\pi^{(i-1)}$. Now by Borovkov [1, p. 13], if in a G/G/1 queue the finite dim distributions of the interarrival and/or service time sequences converge to that of stationary sequences, then the waiting time process (corresponding to our $\{z_k\}$) converges to the same stationary distribution which it would have, if the limiting stationary interarrival and service distributions were used. This implies that if ith queue in $\mathcal{L}^{(i)}$ was fed with $\{x_k(i)\}$ and $\mathbf{1}\{U_k(i) \leqslant q_{\ell(z_k^{(i-1)})}\}$, it would converge to a proper stationary distribution under our assumptions. Since $\{T_k^{(i)}(i)\}$ is a.s. lower bounded by $\mathbf{1}\{U_k(i) \leqslant q_{\ell(z_k^{(i-1)})}\}$, the corresponding limiting distribution of $\{z_k^{(i)}(i)\}$ is also proper. □

The r_1 and r_2 in the above theorem depend only on the arrival rates at the queues. However, $r_i,\ i > 2$, in general, depends on the distributions of the arrival streams, i.e., for the same arrival rates, different distributions of $\{x_k(i)\}$ can provide different values of $r_i,\ i > 2$. This is in sharp contrast to the stability results we find in the queueing literature. This also implies that computing $r_i,\ i > 2$, is not easy. We may need to simulate the system to compute them (as in theorem 2.1). Stability can be checked step by step. Having ensured the stability of $\mathcal{L}^{(i)}$, r_{i+1} can be found by simulating $\mathcal{L}^{(i)}$. Thus this method has an advantage over that of theorem 2.1, because to find $\mathbb{E}_\pi[T(i)]$ directly, it is required to simulate the whole system which may or may not be stable. Thus, the rate of convergence to r_i in simulation and CLT (for confidence interval measurement) can be available a priori under many conditions (see theorem 3.2 and the comments below it), but not for $\mathbb{E}_\pi[T(i)]$.

The above theorem does not provide uniqueness of the stationary distribution. However, if we assume that each arrival stream $\{x_k(i),\ k \geqslant 1\}$ is an aperiodic, regenerative sequence, then under conditions of theorem 3.2 for $s = 1$, we obtain that $\{z_k,\ k \geqslant 0\}$ is aperiodic regenerative with a finite mean cycle length. This provides us with the uniqueness of the stationary distribution and convergence to it in total variation from any initial conditions.

The result below shows that the sufficient condition in theorem 2.3 is quite close to necessary for Markov-modulated arrival streams. Let $\{x_k(l)\}$ be modulated by an irreducible, aperiodic, finite-state Markov chain $\{\gamma_k(l)\}$. Let the modulating chains be mutually independent. Denote

$$\gamma_k = \big(\gamma_k(1), \ldots, \gamma_k(M)\big).$$

Then $\{(z_k, \gamma_k)\}$ is a Markov chain. The proof of the following theorem is obtained using a result from [24] and is given in the appendix A.

Theorem 2.4. Suppose $\{x_k(l)\}$ is Markov-modulated as described above. Then the Markov chain $\mathcal{L}$ is transient if for some j,

$$\lambda_j > r_j. \tag{2.7}$$

Note that the sufficient conditions in theorem 2.3 are not explicit; r_j, $j \geqslant 3$, have to be estimated by simulations. So our next goal is to obtain explicit conditions, though this will, in general, reduce the stability region. We achieve this by defining upper-bounding systems, for which r_i's can be computed explicitly using only the arrival rates at the queues.

Define $\mathcal{M}^{(i)}$, $1 \leqslant i \leqslant M$, as follows. System $\mathcal{M}^{(i)}$ consists of M queues with the same input process as $\mathcal{L}^{(i)}$. We use the *tilde* symbol for quantities concerning $\mathcal{M}^{(i)}$ systems; their meaning remains similar to their counterparts for $\mathcal{L}^{(i)}$. For $\mathcal{M}^{(i)}$ define

$$\begin{aligned} &\tilde{z}_{k+1}^{(i)}(j) = \bigl(\tilde{z}_k^{(i)}(j) - \widetilde{T}_k^{(i)}(j)\bigr)^+ + x_k(j), && 1 \leqslant i \leqslant M-1,\\ &\tilde{z}_k^{(i)}(j) \ \equiv \infty, && i+1 \leqslant j \leqslant M, \end{aligned} \tag{2.8}$$

where for all i, $\widetilde{T}_k^{(i)}(1)$ is an *i.i.d.* sequence of Bernoulli random variables with distribution

$$\mathbb{P}\bigl(\widetilde{T}_k^{(i)}(1) = 1\bigr) = q_M = 1 - \mathbb{P}\bigl(\widetilde{T}_k^{(i)}(1) = 0\bigr).$$

On the other hand, queues $2, \ldots, i$ in each $\mathcal{M}^{(i)}$, are controlled by *independent* systems $\mathcal{M}^{(1)}, \ldots, \mathcal{M}^{(i-1)}$ (these systems are obtained by independent copies of $\{x_k(j)\}$ for each j), respectively, in the following sense: for $2 \leqslant i \leqslant M$, $2 \leqslant j \leqslant i$

$$\mathbb{P}\bigl(\widetilde{T}_k^{(i)}(j) = 1 \bigm| \ell\bigl(\tilde{z}_k^{(j-1)}\bigr) = l\bigr) = q_l = 1 - \mathbb{P}\bigl(\widetilde{T}_k^{(i)}(j) = 0 \bigm| \ell\bigl(\tilde{z}_k^{(j-1)}\bigr) = l\bigr),$$

where $\ell(\tilde{z}_k^{(j-1)}) = \text{cardinality}(\tilde{z}_k^{(j-1)}(m) > 0\colon 1 \leqslant m \leqslant M)$ and $\{\tilde{z}_k^{(j-1)}\}$ is under stationarity.

Note that the first $i+1$ queues of $\mathcal{M}^{(i+1)}$ are mutually independent and the first i of these have identical evolution (in distribution) to the first i of $\mathcal{M}^{(i)}$. This facilitates recursive computation of $\tilde{\rho}_l^{(i)}$ and hence $\tilde{r}_i$'s can be obtained explicitly:

$$\tilde{\rho}_0^{(1)} = \frac{\lambda_1}{q_M}, \qquad \tilde{\rho}_1^{(1)} = 1 - \tilde{\rho}_0^{(1)}, \qquad \tilde{\rho}_l^{(i+1)} = \tilde{\rho}_l^{(i)}\mu_{i+1} + \tilde{\rho}_{l-1}^{(i)}(1 - \mu_{i+1}),$$

where $\mu_i = \lambda_i/\tilde{r}_i$. Also, $\tilde{r}_{i+1} = \mathbb{E}_{\tilde{\pi}^{(i)}}[\widetilde{T}^{(i)}(j)]$, and hence,

$$\tilde{r}_1 = q_M, \qquad \tilde{r}_{i+1} = \sum_{l=0}^{i} \tilde{\rho}_l^{(i)} q_{M-l}, \quad i = 1, 2, \ldots, M-1. \tag{2.9}$$

Observe that $\tilde{r}_1 = r_1 = q_M$ and $\tilde{r}_2 = r_2$, while $\tilde{r}_{i+1}$, $i \geqslant 2$, is a function of $\lambda_1, \lambda_2, \ldots, \lambda_i$, but is insensitive to the detailed distribution of the first i arrival processes, unlike r_{i+1}, $i \geqslant 2$, in theorem 2.3.

In each $\mathcal{M}^{(i)}$ system, for $1 \leqslant j \leqslant i$, using the results for discrete time G/G/1 queue, the jth queue is stable if and only if $\lambda_j < \tilde{r}_j$. Hence, $\mathcal{M}^{(i)}$ is stable if and only if $\lambda_j < \tilde{r}_j$, $1 \leqslant j \leqslant i$.

This leads us to the following theorem.

Theorem 2.5. $\mathcal{L}^{(i)}$ is stable if

$$\lambda_j < \tilde{r}_j, \quad 1 \leqslant j \leqslant i.$$

In particular, $\mathcal{L}$ is stable if

$$\lambda_j < \tilde{r}_j, \quad \text{for all } j.$$

Proof. It is enough to prove that $\mathcal{M}^{(i)}$ stochastically upper-bounds $\mathcal{L}^{(i)}$, for all i. This is trivially true for $i = 1$ since $\mathcal{L}^{(1)}$ and $\mathcal{M}^{(1)}$ are identical systems. To prove the statement for $i = j > 1$, assume it is true for $i \leqslant j - 1$. Denote by $z_k^{(i)}$ and $\tilde{z}_k^{(i)}$, the queue lengths of $\mathcal{L}^{(i)}$ and $\mathcal{M}^{(i)}$, respectively, where $\mathcal{M}^{(1)}, \ldots, \mathcal{M}^{(i-1)}$ are used in the control of $\tilde{z}_k^{(i)}(2)$, $\tilde{z}_k^{(i)}(3), \ldots, \tilde{z}_k^{(i)}(i)$, respectively. By induction hypothesis, we can assume $z_k^{(i)} \leqslant \tilde{z}_k^{(i)}$ a.s. for $i \leqslant j - 1$. From proof of theorem 2.3, we already know that $\mathcal{L}^{(i)}$ stochastically upper bounds $\mathcal{L}^{(j)}$ for $i \leqslant j - 1$, and hence, we can assume $\tilde{z}_k^{(i)} \geqslant z_k^{(j)}$, $k \geqslant 0$, $i \leqslant j - 1$. Note that

$$T_k^{(j)}(l) = \mathbf{1}\{U_k(l) \leqslant q_{\ell(z_k^{(j)})}\} \quad \text{and} \quad \widetilde{T}_k^{(j)}(l) = \mathbf{1}\{U_k(l) \leqslant q_{\ell(z_k^{(l-1)})}\} \quad \text{for } 1 \leqslant l \leqslant j.$$

So if we assume $z_0^{(j)} \leqslant \tilde{z}_0^{(j)}$ a.s., we get $z_k^{(j)} \leqslant \tilde{z}_k^{(j)}$ a.s. for $k \geqslant 1$. This completes the proof. □

The example below supports the finding that the stability region obtained in theorem 2.3 is sensitive to distributions of the arrival process.

Example 2.6. Let $M = 3$, $q_1 = 1$, $q_2 = 0.95$, $q_3 = 0.6$. Consider the following two cases of arrival processes with same arrival rates but different distributions:

$$\begin{aligned} \text{Case 1}: \quad & \mathbb{P}(x(1) = 1) = 0.4 = 1 - \mathbb{P}(x(1) = 0), \\ & \mathbb{P}(x(2) = 1) = 0.45 = 1 - \mathbb{P}(x(2) = 0), \\ \text{Case 2}: \quad & \mathbb{P}(x(1) = 50) = 0.008 = 1 - \mathbb{P}(x(1) = 0), \\ & \text{while } x(2) \text{ has the same distribution as in case 1.} \end{aligned}$$

The arrival rates for both the cases are $\lambda_1 = 0.4$, $\lambda_2 = 0.45$. The calculations show that $\tilde{r}_1 = r_1 = 0.6$, $\tilde{r}_2 = r_2 = 0.72$, $\tilde{r}_3 = 0.809$, which are the same for both. But r_3 computed by simulations is different; in the first case $r_3 = 0.84$, in the second $r_3 = 0.81$.

Note that the difference between r_3 and $\tilde{r}_3$ is quite small in the above example and as a consequence also the differences between the r_3 in the two cases.

Now we identify cases, when $\tilde{r}_n$ is close to r_n, for $n \geqslant 3$. We comment only for $n = 3$; the analysis for $n > 3$ is quite similar. First, suppose $q_1 - q_M$ is small. Then, since $q_1 > r_3 \geqslant \tilde{r}_3 > q_M$, $\tilde{r}_3$ is close to r_3. Next assume $\lambda_2 \downarrow 0$. Then, by our assumption $\lambda_1 \leqslant \lambda_2$, $\lambda_1 \downarrow 0$ also. In this case, both $\tilde{r}_3$ and r_3 tend to q_{M-2} by (2.6) and (2.9), respectively. Thus, under light traffic in the first two queues, $\tilde{r}_3$ is close to r_3. Next suppose $\lambda_1 \uparrow q_M$. Then note that $r_2 \downarrow q_M$ by the explicit equation for r_2 that appears just below (2.6). Also, by assumption, $\lambda_2 \geqslant \lambda_1$, and hence, $r_2 \downarrow \lambda_2$, which is increasing to q_M. Then again by (2.6) and (2.9), $\tilde{r}_3$ and r_3 decrease to q_M. Consequently, even in heavy traffic, both are quite close. Next, let $\lambda_1 \downarrow 0$. Then $r_2 \uparrow q_{M-1}$. Also assume $\lambda_2 \uparrow q_{M-1}$ with $\lambda_2 < r_2$ (we must ensure stability). Then again, $\tilde{r}_3$ and r_3 tend to q_{M-2}. Thus, with queue 1 in light traffic and queue 2 in heavy traffic, $\tilde{r}_3$ and r_3 are quite close to each other. This shows that for the cases considered above, the explicit sufficient conditions of theorem 2.5 are, in fact, quite close to the necessary and sufficient conditions. The following example supports this claim.

Example 2.7. $M = 4$, $q_1 = 1.0$, $q_2 = 0.9$, $q_3 = 0.8$, $q_4 = 0.7$. The arrival process is *i.i.d.* at each node. x takes a value 0, 1 or 10. For all the nodes $\mathbb{P}(x = 10) = 0.02$. Probabilities $\mathbb{P}(x = 0)$ and $\mathbb{P}(x = 1)$ are adjusted so as to achieve the required arrival rate. See table 1 for the results.

Now we discuss the applicability of the above results for slotted-ALOHA protocol without capture. Assume that every node attempts the head-of-the-line packet with probability p, $0 < p < 1$. If i queues are nonempty, the probability of successful packet reception for a packet from any of them is $q_i = p(1 - p)^{i-1}$, $1 \leqslant i \leqslant M$. However, the packet successes in a slot are mutually dependent, since there can not be more than one success (this is possible in a capture channel). Although, this is a different model from the original one, theorems 2.1, 2.3 and 2.4 are still applicable with the above set of q_i's. The Lyones' approach used in the proof of theorem 2.1 can still be applied (in fact, [27] uses the same). Theorem 2.3 uses the fact that $\mathcal{L}^{(i)}$ upperbounds $\mathcal{L}^{(i+1)}$, which continues to hold for the new channel model. If the arrival process is Markov-modulated with the modulating chain vector δ_k, then $\{(z_k, \delta_k)\}$ is a Markov chain. The arguments used in the proof of theorem 2.4 can be applied to this Markov chain in a straightforward way. Hence, even theorem 2.4 holds true. However, theorem 2.5 is not applicable, since $\mathcal{M}^{(i)}$ does not upperbound $\mathcal{L}^{(i)}$. This follows from the fact that in $\mathcal{L}^{(i)}$ at most one packet success is possible in

Table 1

λ_1	λ_2	λ_3	λ_4	r_1	r_2	r_3	$\tilde{r}_3$	r_4	$\tilde{r}_4$
0.2	0.3	0.4	0.5	0.7	0.7714	0.8348	0.8325	0.8904	0.8845
0.3	0.4	0.5	0.6	0.7	0.7571	0.8065	0.8043	0.8486	0.8421
0.4	0.5	0.6	0.7	0.7	0.7429	0.7775	0.7755	0.8040	0.7982
0.69	0.7	0.7	0.7	0.7	0.7014	0.7023	0.7016	0.7030	0.7019
0.69	0.7	0.701	0.702	0.7	0.7014	0.7023	0.7016	0.7029	0.7017

a slot, whereas in $\mathcal{M}^{(i)}$ ($i \geqslant 2$), more than one success is possible, since the first i queues operate mutually independently and each can have a successful transmission with positive probability.

3. Rates of convergence and limit theorems

In this section we obtain rates of convergence to the stationary distribution, finiteness of stationary moments and various other limit theorems. Throughout this section we will assume that $\{x_k(i),\ k \geqslant 0\}$ is a regenerative sequence for each i. Then we will show that the system $\{(z_k(i), x_k(i),\ k \geqslant 0),\ 1 \leqslant i \leqslant M\}$ is regenerative with regeneration length (say) τ. We will prove $\mathbb{E}[\tau^s] < \infty$ for $s \geqslant 1$; then all the results mentioned above will follow as a consequence. To prove this result we will first mention a similar result on a general discrete time regenerative single server queue. This result is proved in [30] and generalizes a result in [29]. We use this result to prove theorem 3.2.

Consider a single server slotted queueing system with the number of arrivals in slot k as a_k and the queue length at time k as q_k. The queue evolves as

$$q_{k+1} = (q_k - b_k)^+ + a_k,$$

where $\{(a_k, b_k),\ k \geqslant 0\}$ is a regenerative sequence with regeneration epochs as $\tau_0 = 0, \tau_1, \tau_2, \ldots$ and $b_k \in \{0, 1\}$. We assume that $\{(a_k, b_k)\}$ is an aperiodic sequence with $\mathbb{E}[\tau_1] < \infty$. Then this sequence has a stationary distribution μ. By Loynes' result [23], $\{q_k\}$ has a unique stationary distribution if

$$\mathbb{E}_\mu[a_k] < \mathbb{E}_\mu[b_k] \tag{3.1}$$

which we assume for the next theorem. Then we have (see [30]):

Theorem 3.1. Assume (3.1),

$$\mathbb{E}\left[\tau_1^s\right] < \infty \quad \text{and} \quad \mathbb{E}\left[\left(\sum_{k=0}^{\tau_1 - 1} a_k\right)^s\right] < \infty \quad \text{for } s \geqslant 1.$$

Then the regeneration length $\hat{\tau}$ (this epoch is defined as a τ_i at which q_k is zero) of $\{(a_k, b_k, q_k)\}$ satisfies $\mathbb{E}[\hat{\tau}^s] < \infty$.

By [15, p. 22], if $\{a_k\}$ are *i.i.d.* and τ_1 is a stopping time then

$$\mathbb{E}\left[\left(\sum_{k=0}^{\tau_1 - 1} a_k\right)^s\right] < \infty \quad \text{if } \mathbb{E}\left[\tau_1^s\right] < \infty \text{ and } \mathbb{E}\left[a_1^s\right] < \infty.$$

For a more general situation, for $p > 1$, $1/p + 1/q = 1$,

$$\mathbb{E}\left[\left(\sum_{k=0}^{\tau_1-1} a_k\right)^s\right] \leqslant \mathbb{E}\left[\tau_1^s\Big(\max_{0\leqslant k\leqslant \tau_1-1} a_k\Big)^s\right] \leqslant \mathbb{E}\left[\tau_1^{sp}\right]^{1/p}\mathbb{E}\Big[\max_{0\leqslant k\leqslant \tau_1-1} a_k^{sq}\Big]^{1/q}$$
$$\leqslant \mathbb{E}\left[\tau_1^{sp}\right]^{1/p}\mathbb{E}\left[\sum_{0\leqslant k\leqslant \tau_1-1} a_k^{sq}\right]^{1/q} = \mathbb{E}\left[\tau_1^{sp}\right]^{1/p}\mathbb{E}[\tau_1]^{1/q}\mathbb{E}_\pi\left[a_1^{sq}\right]^{1/q}.$$

Thus, we need $\mathbb{E}[\tau^{sp}] < \infty$ and $\mathbb{E}_\pi[a_1^{sq}] < \infty$. Frequently optimum conditions will be obtained from the following result of Borovkov and Utev [2]: Let $\{\mathcal{F}_k\}$ be a filteration such that a_k is measurable $\mathcal{F}_k$ and τ_1 is a stopping time with respect to $\{\mathcal{F}_k\}$. Then, if there exist constants A_s and B_s for $s \geqslant 1$, such that

$$\mathbb{E}[a_k^s \mid \mathcal{F}_{k-1}] \leqslant A_s \quad \text{for all } k, \quad \text{and} \quad \mathbb{E}\left[\tau_1^s\right] \leqslant B_s,$$

we obtain $\mathbb{E}[(\sum_{k=1}^{\tau_1} a_k)^s] \leqslant C(A_s B_1 + A_1^s B_s)$, where C is a constant not dependent upon the distributions.

In [30] it is shown that if in addition to conditions in theorem 3.1, we also have $\mathbb{E}[\exp(\gamma \sum_{k=0}^{\tau_1-1} a_k)] < \infty$ for some $\gamma > 0$, then there is $0 < \gamma' \leqslant \gamma$ such that $\mathbb{E}[\exp(\gamma'\tau)] < \infty$.

We use the above theorem to prove $\mathbb{E}[\tau^s] < \infty$ for our system.

Theorem 3.2. Let $\{x_k(i)\}$, $i \geqslant 1$, be aperiodic, regenerative, with regeneration length $\tau(i)$ having finite sth moment, $s \geqslant 1$, and $\{x_k(i)\}$ is independent of $\{x_k(j)\}$, $i \neq j$. Also for $1 \leqslant j \leqslant i$, let $\lambda_j < r_j$, and

$$\mathbb{E}\left[\left(\sum_{k=0}^{\tau(j)-1} x_k(j)\right)^s\right] < \infty.$$

Then the regeneration length $\tilde{\tau}(i)$ of the system $\mathcal{L}^{(i)}$ has finite sth moment. If $i = M$ then we get $\mathbb{E}[\tau^s] < \infty$.

Proof. First consider system $\mathcal{L}^{(1)}$. For this system we can take $\{T_k^{(1)}(1)\}$ to be *i.i.d.* with $\mathbb{P}(T_k^{(1)}(1) = 1) = q_M$. By theorem 3.1, this system is regenerative, aperiodic with $\mathbb{E}[(\tilde{\tau}(1))^s] < \infty$, where $\tilde{\tau}(i)$ is a regeneration length for the system $\mathcal{L}^{(i)}$. Now assume $\mathcal{L}^{(i)}$ is regenerative, aperiodic with $\mathbb{E}[(\tilde{\tau}(i))^s] < \infty$. We will prove $\mathcal{L}^{(i+1)}$ is regenerative with $\mathbb{E}[(\tilde{\tau}(i+1))^s] < \infty$. Observe that since $\{(z_k^{(i)}, x_k(1), \ldots, x_k(i), T_k^{(i)}),\ k \geqslant 0\}$ is regenerative, aperiodic and independent of $\{x_k(i+1)\}$ and each has finite sth moment of regeneration cycles, by Kalashnikov [17, p. 92], we can take $\{(T_k^{(i)}, x_k(1), \ldots, x_k(i+1))\}$ regenerative with finite sth moment. By appropriate construction (as in theorem 2.1), queues in system $\mathcal{L}^{(i+1)}$ are sample pathwise upper-bounded by a system, whose first i queues are generated by $\{(T_k^{(i)}, x_k(1), \ldots, x_k(i))\}$ and $(i+1)$th queue is generated by $\{(\widetilde{T}_k^{(i)}, x_k(i+1))\}$, where $\{\widetilde{T}_k^{(i)}\}$ is an independent copy of any component of $\{T_k^{(i)}\}$. The first i queues of this upper bound system are identical to $\mathcal{L}^{(i)}$, and hence, regenerative, aperiodic with finite sth moment. Denote by $\tilde{\tau}$ a common regeneration length of $\{\tilde{\tau}_k(i),\ k \geqslant 0\}$ and $\{x_k(i+1)\}$, where $\mathbb{E}[\tilde{\tau}^s] < \infty$. It can be written as $\tilde{\tau} = \sum_{k=1}^{\hat{\tau}} \tau_k(i+1)$, where $\hat{\tau}$ is a stopping time and $\{\tau_k(i+1)\}$ are *i.i.d.* with the distribution of $\tau(i+1)$. Also, $\mathbb{E}[\hat{\tau}^s] < \infty$. Then by theorem 3.1, the $(i+1)$th queue is regenerative with finite sth moment if

$$\mathbb{E}\left[\left(\sum_{k=1}^{\hat{\tau}} Y_k\right)^s\right] < \infty,$$

where

$$\{Y_k\} \text{ are } \textit{i.i.d.}, \qquad Y_1 = \sum_{k=1}^{\tau(i+1)} x_k(i+1).$$

By assumption, $\mathbb{E}[Y_1^s] < \infty$. Thus, $\mathbb{E}[\hat{\tau}^s] < \infty$ provides $\mathbb{E}[(\sum_{k=1}^{\hat{\tau}} Y_k)^s] < \infty$. Since the $(i+1)$th queue in the upper bound system is independent of the first i queues, again by Kalashnikov [17], all the first $i+1$ queues have common regeneration points with the sth moment of the regeneration length finite. At such regeneration epochs all the first $i+1$ queues in the upper bound system are empty. Therefore, at these epochs $\mathcal{L}^{(i+1)}$ also has the first $i+1$ queues empty, making such epochs the regeneration epochs of $\mathcal{L}^{(i+1)}$ as well. This concludes the proof. □

The argument to obtain finite exponential moments of $\tilde{\tau}(i)$ from theorem 3.1 is exactly the same.

One practically useful conclusion of the above result is that if $\mathbb{E}[\tau^s] < \infty$, $s > 1$, then we obtain the rate of convergence [28]

$$\mathrm{Sup}_A \left|\mathbb{P}\big((z_k, x_k) \in A\big) - \pi(A)\right| < Ck^{1-s}$$

for some constant C. Also, using [28, theorem 3], we can obtain the rate of convergence of sample mean to the stationary mean. This is useful in obtaining the estimates of r_i in the last section.

We now find conditions for the finiteness of sth stationary moment of the queue length process, i.e., $\mathbb{E}_\pi[(z(i))^s] < \infty$ for $s \geqslant 1$. Note that $\{z_k\}$ is regenerative with stopping time τ. Let $k = 0$ be a regeneration epoch, then since $z_k(i) \leqslant \tau$ for $0 \leqslant k \leqslant \tau$

$$\mathbb{E}_\pi\left[\big(z(i)\big)^s\right]\mathbb{E}[\tau] = \mathbb{E}\left[\sum_1^\tau \big(z_k(i)\big)^s\right] \leqslant \mathbb{E}\left[\tau \cdot (\tau)^s\right].$$

This together with the above theorem gives us the following result.

Proposition 3.3. Suppose $\lambda_i < r_i$, for all i and let the assumptions of theorem 3.2 be satisfied for $s+1$, $s \geqslant 1$. Then $\mathbb{E}_\pi[(z(i))^s] < \infty$, $1 \leqslant i \leqslant M$.

A similar condition for exponential moments is as follows: for $\gamma > 0$

$$\begin{aligned}\mathbb{E}_\pi\bigl[\exp\bigl(\gamma z(i)\bigr)\bigr]\mathbb{E}[\tau] &= \mathbb{E}\left[\sum_{k=0}^{\tau-1}\exp(\gamma z_k) \mid k=0 \text{ a regeneration epoch}\right]\\ &\leqslant \mathbb{E}\bigl[\tau\exp(\gamma\tau)\bigr] \leqslant \mathbb{E}\bigl[\tau^p\bigr]^{1/p}\mathbb{E}\bigl[\exp(\gamma\tau q)\bigr]^{1/q}.\end{aligned}$$

Thus, we need $\mathbb{E}[\exp(\gamma'\tau)] < \infty$ for some $\gamma' > \gamma$.

Various functional limit theorems and strong approximation results for $\{z_k(i), k \geqslant 1\}$ and their empirical processes can be obtained from Sharma [28, theorems 1, 4] by ensuring that for some appropriately chosen $r_1 \geqslant 1$, $r_2 \geqslant 1$

$$\mathbb{E}\left[\left(\sum_{k=1}^{\tau} z_k(i)\right)^{r_1}\right] < \infty \quad \text{and} \quad \mathbb{E}\bigl[\tau^{r_2}\bigr] < \infty.$$

For example, if $r_1 = 2,\ r_2 = 1$, we obtain a functional central limit theorem (FCLT) and a functional law of iterated logarithm (FLIL). If $r_1 > 2,\ r_2 > 3$, we obtain a strong approximation of the appropriately normalized process with a Wiener process. However, if $\mathbb{E}[(\sum_{k=1}^{\tau} z_k)^{r_1}] = \infty$ for $r_1 = 2$ but finite for $r_1 = \alpha$, $1 < \alpha < 2$, then the FCLT and FLIL do not hold; but if $\mathbb{E}[\tau] < \infty$, the appropriately normalized process converges to a stable law with index α. Corresponding functional limit theorem also holds where the limiting process is self-similar with the Hurst parameter $1/\alpha$. This shows long range dependence. These results have implications for the high speed networks in which the traffic shows long range dependence.

Theorem 3.2 and all the other related results do not go through for the slotted-ALOHA model without capture described at the end of previous section. The proof of theorem 3.2 requires that $\mathcal{L}^{(i)}$ be upperbounded by a system of independent queues, which does not hold for pure slotted-ALOHA.

4. Analysis of the delay process

Till now we have been studying the workload process. However, the delay of packets (cells) at the users is the more important process for a user. Now we extend all the results to this process and also provide some realistic upper and lower bounds on the moments of stationary delays.

Let $W_k(j)$ denote the total work in queue j (number of slots required to empty the queue – given z_k, this is a random variable) at the beginning of slot k at the jth node and let $W_k = (W_k(1), \ldots, W_k(M))$. We first prove that $\{W_k\}$ has a proper stationary distribution. Note that

$$W_k(j) = \min\left\{n\colon \sum_{l=0}^{n-1} T_{k+l}(j) \geqslant z_k(j)\right\}.$$

Consider a time stationary version of $\{(z_k, T_k, x_k)\}$. The fact that $W_k = f(z_k, T_k, z_{k+1}, T_{k+1}, \ldots)$, for an appropriately defined function f, we obtain stationarity of $\{(W_k, z_k, T_k, x_k)\}$. Properness of the stationary distribution of $\{W_k\}$ follows from the fact that $\mathbb{E}_\pi[T_k] \geqslant q_M$ and hence, $\sum_{l=1}^{n} T_{k+l} \to \infty$ a.s. as $n \to \infty$. The customer stationary version of $\{(W_k, z_k, T_k, x_k)\}$, denoted by $\{(\widehat{W}_k, \hat{z}_k, \widehat{T}_k, \hat{x}_k)\}$, is also proper. Now we show stationarity of the delay process. Let $\widetilde{D}_l(i)$ denote the delay (including the service time) of the first packet in the lth batch that arrives at node i. First we will prove the stationarity of $\{\widetilde{D}_l(i),\ l \geqslant 0\}$ for a fixed i. Let $a_l(i)$ denote the arrival epoch of the lth batch at node i. Then $\{(W_k, z_k, T_k, x_k),\ k \geqslant 0\}$ is a process with an embedded marked point process $\{(a_k(i)),\ k \geqslant 0\}$ (the former is referred to as PMP). Let $K_l(i) = \{(W_k, z_k, T_k, x_k),\ k \geqslant a_l(i)\}$. Then the random marked point process (RMPP) corresponding to the above PMP can be taken as $\{(a_l(i), K_l(i)),\ l \geqslant 0\}$, where $K_l(i)$ serves as the mark of the lth batch arrival. Also, RMPP is stationary if the corresponding PMP is stationary by [10, definition 1.5.2]. So by stationarity of the RMPP, a mark $K_l(i)$ has the same distribution for any l. Also note that $\widetilde{D}_l(i)$ depends only on $K_l(i)$. Hence, we have $(\widetilde{D}_1(i), \widetilde{D}_2(i), \ldots) \overset{d}{=} (\widetilde{D}_2(i), \widetilde{D}_3(i), \ldots)$, which proves the stationarity of $\{\widetilde{D}_l(i)\}$. Let $D_k(i)$ denote the delay of the kth packet that arrives at node i. From the stationarity of $\{\widetilde{D}_l(i)\}$, we get the stationarity of $\{D_k(i)\}$ using a result in [4].

In the rest of the section we make the assumptions in theorem 3.2. Let $z_0 = 0$; then the total number of arrivals in $[0, \tau - 1]$ at node i is $\tau^*(i) = \sum_{k=0}^{\tau-1} x_k(i) \leqslant \tau$. Hence, the sufficient conditions for $\mathbb{E}[\tau^s] < \infty$ also apply to $\mathbb{E}[(\tau^*(i))^s] < \infty$. The following discussion is for a fixed i, so we drop the index i for convenience.

Now we prove finiteness of sth stationary moment of delay D for $s \geqslant 1$. Note that τ^* is a stopping time for D_l. Then

$$\mathbb{E}\left[\sum_{l=1}^{\tau^*} (D_l)^s\right] = \mathbb{E}\big[\tau^*\big]\mathbb{E}_\pi\big[(D_l)^s\big]$$

implies that it is enough to ensure that $\mathbb{E}[\sum_{l=1}^{\tau^*} (D_l)^s] < \infty$. But

$$\mathbb{E}\left[\sum_{l=1}^{\tau^*} (D_l)^s\right] \leqslant \mathbb{E}\left[\sum_{l=1}^{\tau^*} (\tau)^s\right] = \mathbb{E}\big[\tau^* \cdot (\tau)^s\big] \leqslant \mathbb{E}\big[\tau \cdot \tau^s\big] = \mathbb{E}\big[\tau^{s+1}\big].$$

Hence, $\mathbb{E}[(\tau)^{s+1}] < \infty$ will provide $\mathbb{E}_\pi[(D_l)^s] < \infty$.

Next we provide bounds on the moments of the stationary delay when $\{x_k(i)\}$ are *i.i.d.*

Theorem 4.1. For an *i.i.d.* arrival stream $\{x_k(i)\}$ for which $\lambda_j < r_j$ and $\mathbb{E}[(x_1(j))^2] < \infty$ for $j \leqslant i$,

$$\frac{\lambda_i(1 - q_1) + \mathrm{var}(x(i))}{2\lambda_i(q_1 - \lambda_i)} + \frac{1}{2} \leqslant \mathbb{E}_\pi\big[D(i)\big] \leqslant \frac{\lambda_i(1 - q_M) + \mathrm{var}(x(i))}{2\lambda_i(q_M - \lambda_i)} + \frac{1}{2}. \tag{4.1}$$

Proof. Note that the ith queue in $\mathcal{L}^{(i)}$ is upper-bounded by a queue where $\{T_k\}$ are *i.i.d.* with $\mathbb{P}(T_k = 1) = q_M$ and lower bounded by a queue with $\mathbb{P}(T_k = 1) = q_1$. This fact will provide the upper and lower bounds from the following discussion. We show it for the upper bound only. The workload process of this upper bound queue satisfies

$$W_{k+1}(i) = \bigl(W_k(i) - 1\bigr)^+ + \sum_{l=1}^{x_k(i)} S_l,$$

where S_l is *i.i.d.* with Geom(q_M) distribution. Denote by $Y_k(i) = \sum_{l=1}^{x_k(i)} S_l$. From a result in [11], we have ($\bar{\pi}$ denotes the stationary distribution of $\{W_k\}$)

$$\mathbb{E}_{\bar{\pi}}\bigl[W(i)\bigr] = \frac{\text{var}(Y(i))}{2(1 - \mathbb{E}[Y(i)])} + \frac{1}{2}\mathbb{E}\bigl[Y(i)\bigr].$$

Now applying Little's law, we get

$$\mathbb{E}_{\bar{\pi}}\bigl[D(i)\bigr] = \frac{\mathbb{E}_{\bar{\pi}}[W(i)]}{\mathbb{E}[Y(i)]}.$$

Since $\text{var}(Y(i)) = \mathbb{E}[x(i)]\text{var}(S_l) + (\mathbb{E}[S_l])^2\text{var}(x(i))$, $\text{var}(S_l) = (1 - q_M)/q_M^2$ and $\mathbb{E}[S_l] = (1)/(q_M)$, we get the upper bound on $\mathbb{E}_{\pi}[D(i)]$. Repeating the argument with q_M replaced by q_1 provides the lower bound. □

The upper bound in the above theorem of course assumes that $\lambda_i < q_M$. To get a feel for the tightness of these bounds, taking the difference of the upper and lower bounds we get

$$\frac{(q_1 - q_M)(1 - \lambda_i + \text{var}(x(i))/\lambda_i)}{2(q_M - \lambda_i)(q_1 - \lambda_i)}.$$

Thus, we can conclude that as $q_M \uparrow q_1$, this goes to zero and increasing var($x(i)$) or reducing ($q_M - \lambda_i$) increases this. In fact, we can show that as $q_M \uparrow q_1$, the transient as well as stationary distributions of the upper bound system converge stochastically monotonically (and, hence, also all the moments) to that of the lower bound system.

Bounds on the second moment can be similarly obtained using (see [11])

$$\begin{aligned}\mathbb{E}_{\bar{\pi}}\bigl[W^2\bigr] = \Biggl[&\mathbb{E}\bigl[Y^3\bigr] + 3\mathbb{E}_{\bar{\pi}}[W]\frac{1 + \mathbb{E}[Y^2]}{3(1 - \mathbb{E}[Y])}\\ &-\mathbb{E}[Y]\bigl(1 + 6\mathbb{E}[W]\bigr) + 3\mathbb{E}[Y]\bigl(\mathbb{E}[Y] - \mathbb{E}\bigl[Y^2\bigr]\bigr)\Biggr].\end{aligned}$$

If the arrival process $\{x_k(i)\}$ is not *i.i.d.* but modulated by a finite state chain, then closed form expressions for the above bounds are not available. However, algorithms can be developed for such bounds. If $\{x_k(i)\}$ is general stationary ergodic, then the Kingman's heavy traffic approximations (from [34], now these are only approximations, not upper bounds) can be used as approximations for upper and lower bounds.

It is possible to get some asymptotic results on the tails of the stationary distributions. Some of the results of [32, theorem 5] directly carry over using theorem 3.2, while the others can be used on the upper and lower bound systems of theorem 4.1 to get the corresponding bounds.

Now we give two examples of different sets of q_is, each with three users, having Bernoulli arrival process. The examples show that the mean sojourn time bounds obtained by us can be very tight in many cases (when q_M is close to q_1) and quite loose in some other cases (when q_M is much smaller than q_1 and the arrival rates close to q_M). We also observe that in each case r_3 is close to $\tilde{r}_3$ (thus the tightness of our explicit sufficient conditions for stability).

Example 4.2. Consider $M = 3$, each with a Bernoulli traffic with $\mathbb{E}[x_1(i)] = \lambda_i$ and $q_1 = 1.0$, $q_2 = 0.99$, $q_3 = 0.98$. See table 2.

Since the arrival process to each node is Bernoulli and $q_1 = 1.0$, the lower bound on the delay is trivially 1. Hence, we have mentioned only the upper bounds in table 3.

Example 4.3. Let $M = 3$, $x_k(i)$ is Bernoulli and $q_1 = 1.0$, $q_2 = 0.90$, $q_3 = 0.62$. (See table 4.) Since the arrival process to each node is Bernoulli and $q_1 = 1.0$, the lower bound on the delay is trivially 1. Hence, we have mentioned only the upper bounds in table 5.

Table 2

λ_1	λ_2	λ_3	r_1	r_2	r_3	$\tilde{r}_3$
0.4	0.5	0.6	0.98	0.9859	0.9908	0.9908
0.5	0.6	0.7	0.98	0.9849	0.9888	0.9888
0.6	0.7	0.8	0.98	0.9839	0.9868	0.9868
0.7	0.8	0.9	0.98	0.9829	0.9847	0.9847

Table 3

Lambdas			Simulated delay			Upper bounds on the delay[1]		
λ_1	λ_2	λ_3	1	2	3	1	2	3
0.4	0.5	0.6	1.0194	1.0227	1.0248	1.0345 1.48	1.0417 1.85	1.0526 2.72
0.5	0.6	0.7	1.0277	1.0330	1.0386	1.0417 1.36	1.0526 1.90	1.0714 3.16
0.6	0.7	0.8	1.0409	1.0518	1.0709	1.0526 1.13	1.0714 1.86	1.1111 3.76
0.7	0.8	0.9	1.0613	1.0887	1.1799	1.0714 0.96	1.1111 2.06	1.2500 5.94

[1] The second line in each row shows percentage difference between the simulated delay and the upper bound.

Table 4

λ_1	λ_2	λ_3	r_1	r_2	r_3	$\tilde{r}_3$
0.1	0.2	0.3	0.62	0.8548	0.9591	0.9537
0.2	0.3	0.4	0.62	0.8097	0.9198	0.9092
0.3	0.4	0.5	0.62	0.7645	0.8685	0.8537
0.4	0.5	0.6	0.62	0.7194	0.8010	0.7853

Table 5

Lambdas			Simulated delay			Upper bounds on the delay[1]		
λ_1	λ_2	λ_3	1	2	3	1	2	3
0.1	0.2	0.3	1.0821	1.0693	1.0620	1.7308 59.94	1.9048 78.13	2.1875 105.99
0.2	0.3	0.4	1.1604	1.1513	1.1448	1.9048 64.15	2.1875 90.00	2.7273 138.23
0.3	0.4	0.5	1.3210	1.3149	1.3376	2.1875 66.72	2.7273 107.42	4.1667 211.49
0.4	0.5	0.6	1.6846	1.7831	1.9956	2.7273 61.89	4.1667 133.68	20.00 902.21

[1] The second line in each row shows percentage difference between the simulated delay and the upper bound.

Theorem 4.1 goes through for slotted-ALOHA without capture, since the systems operating at service rate q_1 and q_M are still lower-bounding and upper-bounding, respectively.

5. Transient analysis

For data traffic, marginal stationary distributions of delays and probability of packet loss can be sufficient. But for voice and video, the quality of received signal can significantly depend on the transient and joint stationary distributions of the delays and probability of packet loss. Therefore, recently, such performance indices are being increasingly studied (see [5]). In [31,32], we have obtained several such results for single and multiserver queues when the service time is one slot. In the following we extend some of those results to the present system.

For a given $m > 0$, let T_m be the first time a $z_k(i)$ exceeds m in a particular queue i of our system. This will also be the first time the queue i overflows if it had a buffer of length m. For *i.i.d.* $\{T_k(i)\}$ and $\{x_k(i)\}$, we can actually compute the moments and distributions of T_m, for any initial queue length, as in [31] (there $T_k = 1$, but the required changes can be easily made as in the last paragraph of this section). By using upper and lower bound systems of theorem 4.1, these will provide bounds for T_m in our system.

Now we provide an asymptotic result. The initial queue length will not matter. We assume that $\{x_k(j)\}$ are regenerative for $j \leqslant i$, satisfying the assumptions of theorem 3.2 for $s = 1$. Then, in particular, $\mathbb{E}_\pi[x_k(i)] < \infty$ and, hence, $T_m \to \infty$ a.s. as $m \to \infty$. Since the convergence is monotonic, $\mathbb{E}[T_m] \uparrow \infty$. Thus, from [18], $T_m/\mathbb{E}[T_m] \overset{w}{\to}$ exp(1), where $\overset{w}{\to}$ denotes convergence in distribution and exp(1) is exponential distribution with mean 1. Let p_m denote the probability that $z_k(i)$ exceeds m in one regeneration cycle. Then we also have $p_m \mathbb{E}[T_m]/\mathbb{E}[\tau] \to 1$ and, hence,

$$\frac{T_m p_m}{\mathbb{E}[\tau]} \overset{w}{\to} \exp(1). \tag{5.1}$$

It is possible to obtain rates of convergence and bounds on errors in this asymptotic result. To use the above result effectively, we need information on p_m. Below we will provide algorithms to compute upper and lower bounds on p_m. If we make the stronger assumptions that the assumptions of theorem 3.2 are satisfied for $s \geqslant 1$ for system $\mathcal{L}^{(i)}$, then we obtain directly the following results. Denoting the regeneration length of system $\mathcal{L}^{(i)}$ by τ, we then have $\mathbb{E}[\tau^s] < \infty$. This implies that $m^s \mathbb{P}(\tau \geqslant m) \to 0$ as $m \to \infty$. Then, $p_m \leqslant \mathbb{P}(\tau \geqslant m)$ implies that $m^s p_m \to 0$. This of course implies (from the relation mentioned above) that $\mathbb{E}[T_m]/m^s \to \infty$ as $m \to \infty$ and (using Jensen's inequality) for $\beta > 1$, $\mathbb{E}[T_m^\beta] m^{-s\beta} \to \infty$. Also using (5.1) we obtain $\mathbb{P}(T_m > x m^s \mathbb{E}[\tau]) \to 1$ for any $x > 0$. Using the upper bound system, we can also obtain bounds on the exponential decay rate of T_m. All the above results would hold if T_m denotes the first time *any* queue in the system exceeds m. The arguments and the assumptions remain the same.

Now we consider the joint distributions of the epochs when queue i exceeds level m. As in [31, theorem 3(a)], if $N^{(m)}$ is the process which denotes the epochs when queue i exceeds m for the *first time in a regeneration cycle*, then $N^{(m)} p_m/\mathbb{E}[\tau]$ converges in distribution to a Poisson process with rate one. Similarly, conditions can be obtained (assuming *i.i.d.* arrival stream) to show that for $\widetilde{N}^{(m)}$, the process of epochs of exceeding of m by queue i, $\widetilde{N}^{(m)} p_m/\mathbb{E}[\tau]$ converges to a compound Poisson process with a proper distribution of the limiting batch sizes. The results in [31], on the qualitative differences in the asymptotic behaviour of the process of epochs of exceeding of a level m, in a system with infinite buffers studied above and the epochs of packet loss in a finite buffer system, with buffer size m continue to hold. One can also obtain various limit theorems on the process $\{N(n)\}$, where $N(n)$ denotes the number of slots till time n, when $z_k(i)$ exceeds a fixed level m (or the number of packets lost till time n). The details and the arguments remain as in [31].

The following related problem is also of substantial theoretical and practical interest. Let $\tau_j^{(i)}$ be the first time queue i becomes empty if $z_0(i) = j$. For this we can only provide the results based on the upper and lower bound systems defined in theorem 4.1 (which will respectively provide the upper bound and the lower bound for $\tau_j^{(i)}$). We will also assume that $\{x_k(i)\}$ is *i.i.d.* We provide the results for a discrete queue with $\{T_k\}$ *i.i.d.* with $\mathbb{P}(T_k = 1) = q = 1 - \mathbb{P}(T_k = 0)$ and $\{x_k\}$ *i.i.d.* Then the work load process for this system, as obtained in theorem 4.1, is

$$W_{k+1} = (W_k - 1)^+ + \sum_{n=1}^{x_k} s_n,$$

where $\{s_n\}$ are *i.i.d.* (defined for each slot separately) with $\mathbb{P}(s_n = m) = (1 - q)^{m-1}q,\ m \geqslant 1$. Denoting $\sum_{n=1}^{x_k} s_n$ by Y_k, and taking $W_0 = 0$, if $\tau(q)$ denotes the first time $W_k = 0$ for $k > 0$, then

$$\sum_{k=0}^{\tau(q)-1} Y_k = \tau(q) - 1 \tag{5.2}$$

and, hence, $\mathbb{E}[\tau(q)] = 1/(1 - \mathbb{E}[Y]) = 1/(1 - \mathbb{E}[x_1]\mathbb{E}[s_1])$. Taking $q = q_1$ and q_M provides lower and upper bounds on $\mathbb{E}[\tau_0^{(i)}]$. Of course, from (5.2), we can get bounds on higher moments of $\tau_0^{(i)}$ also. To get bounds on moments of $\tau_j^{(i)}$, $j > 0$, observe that if there are j packets in the above *i.i.d.* queue at time $k = 0$, this amounts to a work of $\sum_{k=1}^{j} s_k \triangleq Y$ slots and, hence,

$$\mathbb{E}\big[\tau_j^{(i)}(q)\big] = \sum_{k=j}^{\infty} \frac{k}{q}\mathbb{E}\big[\tau(q)\big]\mathbb{P}(Y = k).$$

6. Markov-modulated capture channel

In this section, we consider a generalization of the original channel model that we have been dealing with so far, which is motivated by the following fact. For a typical wireless channel, it has been observed that a transmission from a user can undergo periods of deep fades (very high attenuation) and these periods could last for several link layer packets. Thus in practice, the channel state in successive slots is certainly dependent. This dependence can be modelled by a Markov chain (more general fading process in a related model is studied in [33]). So we consider the following generalization. The channel state is modulated by a Markov chain, and the packet success probability depends both on the state of the channel and the number of transmitted packets in a slot. Two cases are considered. In the first case, there is a common Markov chain for all the users. This case can be applied when the transmitting users are close to each other, and hence can be assumed to undergo the same fading. In the second case, each user has his own Markov chain, and the Markov chains are mutually independent and identically distributed. This case can be applied when the users are moving independently of each other, but in an environment which is spatially homogeneous. The latter assumption leads to identical fading distributions for all of them. The following two subsections provide the details.

6.1. Common Markov chain

The channel is assumed to be in any one of the L states $\{1, 2, \ldots, L\}$. The state of the channel during slot k is given by δ_k, where $\{\delta_k\}$ is an irreducible, aperiodic discrete-time Markov chain with the state space $\{1, 2, \ldots, L\}$. We assume that the transition probability matrix of the chain is $P = \{(p_{ij})\}$, and its stationary probability vector is $\nu = \{\nu_l,\ l = 1, \ldots, L\}$.

If j packets are transmitted in a slot, and the channel state is l during that slot, a packet is successfully received with probability $q_j^{(l)}$ independently of anything else. A higher channel state l is assumed to have lower attenuation. Also, for a fixed l, the total interference from other users increases as j increases. Hence, the assumptions made on $q_j^{(l)}$'s are

$$\text{for every } j, \quad 0 < q_j^{(1)} \leqslant q_j^{(2)} \leqslant \cdots \leqslant q_j^{(L)}, \tag{6.1}$$

$$\text{for every } l, \quad q_1^{(l)} \geqslant q_2^{(l)} \geqslant \cdots \geqslant q_M^{(l)} > 0. \tag{6.2}$$

All other assumptions in the original model are still valid. The system can be described by the equation

$$z_{k+1}(j) = \bigl(z_k(j) - T_k(j)\bigr)^+ + x_k(j). \tag{6.3}$$

The above assumptions imply that, for $1 \leqslant i \leqslant M,\ 1 \leqslant j \leqslant M$,

$$\mathbb{P}\bigl(T_k(j) = 1 \mid l(z_k) = i,\ \delta_k = l\bigr) = q_i^{(l)},$$
$$\mathbb{P}\bigl(T_k(j) = 0 \mid l(z_k) = i,\ \delta_k = l\bigr) = 1 - q_i^{(l)}.$$

Also, given (z_k, δ_k), $\{T_k(j),\ 1 \leqslant j \leqslant M\}$ are mutually independent and T_k is independent of $T_{k-1}, T_{k-2}, \ldots, T_1$.

Theorem 2.1 for the original system has its counterpart for this system which has a similar proof.

Theorem 6.1. Suppose $\{x_k\}$ is stationary. Then there exists a stationary solution $\{z_k\}$ (with distribution π) and starting from $z_0 \equiv 0$, z_k converges weakly to it. If $\{x_k\}$ is also ergodic, and ($T(i)$ is a generic *r.v.* of sequence $\{T_k(i),\ k \geqslant 1\}$)

$$\lambda_i < \mathbb{E}_\pi\bigl[T(i)\bigr], \quad 1 \leqslant i \leqslant M, \tag{6.4}$$

then the distribution π of this stationary solution is proper.

Proof. It can be shown that equation (6.3) can be written in the form

$$z_{k+1} = g\bigl(z_k, (U_k, x_k, \delta_k)\bigr),$$

where g is monotonically increasing in the first component and $\{(U_k, x_k, \delta_k)\}$ is a stationary sequence. The rest of the proof is the same as that of theorem 2.1. $\square$

Define $q_i = \sum_{l=1}^{L} \nu_l q_i^{(l)}$. The service rate $\mathbb{E}_\pi[T(i)]$ for a user under stationarity is lower bounded by q_M, which is the service rate in a system in which all the M users are always transmitting. This leads to the following corollary.

Corollary 6.2. The system is stable if

$$\lambda_i < q_M, \quad 1 \leqslant i \leqslant M. \tag{6.5}$$

System $\mathcal{L}^{(i)}$ is defined in exactly the same way as before – set queue $i+1$ onwards to infinity (which are then always transmitting) and allow the first i queues to evolve under the same channel model. Let

$$\rho_{j,l}^{(i)} = \mathbb{P}_{\pi^{(i)}}\left(\begin{array}{l}\text{exactly } j \text{ queues are empty among the first } i \text{ queues}\\ \text{in slot } k \text{ and } \delta_k = l\end{array}\right).$$

Define

$$r_1 = \sum_{l=1}^{L} \nu_l \, q_M^{(l)}.$$

For $i \geqslant 2$, define

$$r_{i+1} = \sum_{j=0}^{i} \sum_{l=1}^{L} \rho_{j,l}^{(i)} \, q_{M-j}^{(l)}.$$

The following is the counterpart of theorem 2.3, and has a similar proof.

Theorem 6.3. If

$$\lambda_j < r_j, \quad 1 \leqslant j \leqslant i,$$

then $\mathcal{L}^{(i)}$ is stable. In particular, $\mathcal{L}$ is stable if

$$\lambda_j < r_j, \quad 1 \leqslant j \leqslant M.$$

Proof. From corollary 6.2, we know that $\mathcal{L}^{(1)}$ is stable if $\lambda_1 < r_1 = q_M$. The proof of the fact that the stability of $\mathcal{L}^{(i-1)}$ and $\lambda_i < r_i$ imply the stability of $\mathcal{L}^{(i)}$, is similar to the proof of this same statement in theorem 2.3. In particular, note that $\{T_k^{(i)}(i)\}$ are a.s. lower bounded by sequence

$$\mathbf{1}\big\{U_k(i) \leqslant q_{\ell(z_k^{(i-1)})}^{(\delta_k)}\big\},$$

which is converging weakly to $\{T_k^{(i-1)}(j)\}$ under the stationary distribution $\pi^{(i-1)}$. We drop the rest of the details. □

The argument for the stationarity of the work process $\{W_k\}$ and the delay process $\{D_k\}$ is similar to that provided in the second paragraph of section 4. In particular, W_k can be expressed by $W_k = f(z_k, T_k, \delta_k, z_{k+1}, T_{k+1}, \delta_{k+1}, \ldots)$, for an appropriately defined f, which proves the fact that $\{W_k\}$ is stationary. Also the observation that $\{(W_k, z_k, T_k, x_k, \delta_k),\ k \geqslant 0\}$ is a process with an embedded marked point process $\{(a_k(i)), k \geqslant 0\}$ can be used to prove the stationarity of $\{D_k\}$.

Let $\{x_k(l)\}$ be modulated by an irreducible, aperiodic, finite-state Markov chain $\{\gamma_k(l)\}$. Let the modulating chains be mutually independent.

Denote $\gamma_k = (\gamma_k(1), \ldots, \gamma_k(M))$. Then $\{(z_k, \gamma_k, \delta_k)\}$ is a Markov chain. The proof of the following theorem is given in appendix A.

Theorem 6.4. Suppose $\{x_k(l)\}$ is Markov-modulated for each l. Then the Markov chain $\mathcal{L}$ is transient if for some j,

$$\lambda_j > r_j. \tag{6.6}$$

The counterpart of theorem 2.5 does not exist in this case, since this system can not be upper-bounded by a system of independent queues as in the previous case. The modulating Markov chain, which is common for all the nodes, does not allow such upper-bounding. For the same reason, even the counterpart of theorem 3.2 does not exist. However, we will show that these results extend to the independent fading chain case.

Under the conditions of corollary in this section, delay bounds are available for Markov-modulated arrival processes. Consider a single queue described by

$$c_{k+1} = (c_k - b_k)^+ + a_k,$$

where a_k, b_k, c_k denote the number of arrivals during slot k, the number of departures during slot k (which could be 0 or 1) and the queue length at the beginning of slot k respectively. Let both $\{a_k\}$ and $\{b_k\}$ be Markov modulated; $\{b_k\}$ is Markov modulated by an L-state chain $\{d_k\}$, such that

$$\mathbb{P}(b_k = 1 \mid d_k = l) = q^{(l)} = 1 - \mathbb{P}(b_k = 0 \mid d_k = l).$$

Approximations of sojourn time moments are available for such a queue (see [8,35,37]). Denote the ith moment of sojourn time by $S^{(i)}(\{a_k\}, \{(d_k), q^{(1)}, \ldots, q^{(L)}\})$. Then the ith moment of sojourn time for queue j in our system satisfies

$$\begin{aligned} S^{(i)}\big(\{x_k(j)\}, \{(\delta_k), q_1^{(1)}, \ldots, q_1^{(L)}\}\big) &\leqslant \mathbb{E}_\pi\big[D^i(j)\big] \\ &\leqslant S^{(i)}\big(\{x_k(j)\}, \{(\delta_k), q_M^{(1)}, \ldots, q_M^{(L)}\}\big). \end{aligned}$$

Simulation examples are given at the end of the section.

6.2. *Independent Markov chains*

In this section, we consider a generalization of the original channel model in which different nodes have their own channel states in the same slot.

The channel is assumed to be in any one of the L states $\{1, 2, \ldots, L\}$. The state of the channel during slot k for node i is given by $\delta_k(i)$, where each $\{\delta_k(i)\}$ is an irreducible, aperiodic discrete-time Markov chain with the state space $\{1, 2, \ldots, L\}$. Also the chains are mutually independent of each other and have the same distribution. We assume that the transition probability matrix of the chain is $P = \{(p_{ij})\}$, and its stationary probability vector is $\nu = \{\nu_l,\ l = 1, \ldots, L\}$.

If j packets are transmitted in a slot, and the channel state for node i is l during that slot, then a packet transmitted by node i is successfully received with probability $q_j^{(l)}$ independently of anything else. The assumptions made on $q_j^{(l)}$'s are (same as in previous section)

$$\text{for every } j, \quad 0 < q_j^{(1)} \leqslant q_j^{(2)} \leqslant \cdots \leqslant q_j^{(L)}, \tag{6.7}$$

$$\text{for every } l, \quad q_1^{(l)} \geqslant q_2^{(l)} \geqslant \cdots \geqslant q_M^{(l)} > 0. \tag{6.8}$$

All other assumptions in the original model are still valid. The system can be described by the equation

$$z_{k+1}(j) = \bigl(z_k(j) - T_k(j)\bigr)^+ + x_k(j). \tag{6.9}$$

The above assumptions imply that, for $1 \leqslant i \leqslant M$, $1 \leqslant j \leqslant M$,

$$\mathbb{P}\bigl(T_k(j) = 1 \mid l(z_k) = i,\ \delta_k(j) = l\bigr) = q_i^{(l)},$$
$$\mathbb{P}\bigl(T_k(j) = 0 \mid l(z_k) = i,\ \delta_k(j) = l\bigr) = 1 - q_i^{(l)}.$$

Also, given (z_k, δ_k), $\{T_k(j),\ 1 \leqslant j \leqslant M\}$ are mutually independent and T_k is independent of $T_{k-1}, T_{k-2}, \ldots, T_1$.

Theorem 2.1 for the original system has its counterpart for this system which has a similar proof.

Theorem 6.5. Suppose $\{x_k\}$ is stationary. Then there exists a stationary solution $\{z_k\}$ (with distribution π) and starting from $z_0 \equiv 0$, z_k converges weakly to it. If $\{x_k\}$ is also ergodic, and ($T(i)$ is a generic *r.v.* of sequence $\{T_k(i),\ k \geqslant 1\}$)

$$\lambda_i < \mathbb{E}_\pi\bigl[T(i)\bigr], \quad 1 \leqslant i \leqslant M, \tag{6.10}$$

then the distribution π of this stationary solution is proper.

Proof. It can be shown that equation (6.9) can be written in the form

$$z_{k+1} = g\bigl(z_k, (U_k, x_k, \delta_k)\bigr).$$

The rest of the proof remains the same. □

Define $q_i = \sum_{l=1}^{L} \nu_l q_i^{(l)}$. Then corollary 2.2 goes through. In fact, all the other results also go through with this new set of q_i's.

Corollary 6.6. The system is stable if

$$\lambda_i < q_M, \quad 1 \leqslant i \leqslant M. \tag{6.11}$$

System $\mathcal{L}^{(i)}$ is defined in exactly the same way – set queue $i+1$ onwards to infinity (which are then always transmitting) and allow the first i queues to evolve under the same channel model. Let

$$\rho_j^{(i)} = \mathbb{P}_{\pi^{(i)}}(\text{exactly } j \text{ queues are empty among the first } i \text{ queues}).$$

Define

$$r_1 = \sum_{l=1}^{L} \nu_l q_M^{(l)} = q_M.$$

For $i \geqslant 2$, define

$$r_{i+1} = \sum_{j=0}^{i} \sum_{l=1}^{L} \rho_j^{(i)} \nu_l q_{M-j}^{(l)} = \sum_{j=0}^{i} \rho_j^{(i)} q_{M-j}.$$

Theorem 2.3 goes through.

Theorem 6.7. If

$$\lambda_j < r_j, \quad 1 \leqslant j \leqslant i,$$

then $\mathcal{L}^{(i)}$ is stable. In particular, $\mathcal{L}$ is stable if

$$\lambda_j < r_j, \quad 1 \leqslant j \leqslant M.$$

Let $\{x_k(l)\}$ be modulated by an irreducible, aperiodic, finite-state Markov chain $\{\gamma_k(l)\}$. Let the modulating chains be mutually independent. Denote $\gamma_k = (\gamma_k(1), \ldots, \gamma_k(M))$. Then $\{(z_k, \gamma_k, \delta_k)\}$ is a Markov chain. The proof of the following theorem is given in appendix A.

Theorem 6.8. Suppose $\{x_k(l)\}$ is Markov-modulated for each l. Then the Markov chain $\mathcal{L}$ is transient if for some j,

$$\lambda_j > r_j. \tag{6.12}$$

Systems $\mathcal{M}^{(i)}$ can be defined as before. System $\mathcal{M}^{(i)}$ consists of M queues with the same input process as $\mathcal{L}^{(i)}$. We use the *tilde* symbol for quantities concerning $\mathcal{M}^{(i)}$ systems, their meaning remains similar to their counterparts for $\mathcal{L}^{(i)}$. For $\mathcal{M}^{(i)}$ define

$$\begin{aligned} \tilde{z}_{k+1}^{(i)}(j) &= \bigl(\tilde{z}_k^{(i)}(j) - \widetilde{T}_k^{(i)}(j)\bigr)^+ + x_k(j), \quad && 1 \leqslant i \leqslant M-1, \\ \tilde{z}_k^{(i)}(j) &\equiv \infty, && i+1 \leqslant j \leqslant M, \end{aligned} \tag{6.13}$$

where for all i, $\widetilde{T}_k^{(i)}(1)$ is a Markov modulated sequence modulated by the chain $\{\delta_k(1)\}$ and has a distribution

$$\mathbb{P}\bigl(\widetilde{T}_k^{(i)}(1) = 1 \mid \delta_k(1) = l\bigr) = q_M^{(l)} = 1 - \mathbb{P}\bigl(\widetilde{T}_k^{(i)}(1) = 0 \mid \delta_k(1) = l\bigr).$$

On the other hand, queues $2, \dots, i$ in each $\mathcal{M}^{(i)}$, are controlled by *independent* systems $\mathcal{M}^{(1)}, \dots, \mathcal{M}^{(i-1)}$ (these systems are obtained by independent copies of $\{x_k(j)\}$ for each j), respectively, in the following sense: for $2 \leqslant i \leqslant M$, $2 \leqslant j \leqslant i$,

$$\begin{aligned} &\mathbb{P}\big(\widetilde{T}_k^{(i)}(j) = 1 \mid \ell\big(\tilde{z}_k^{(j-1)}\big) = l, \delta_k(j) = r\big) \\ &\quad = q_l^{(r)} = 1 - \mathbb{P}\big(\widetilde{T}_k^{(i)}(j) = 0 \mid \ell\big(\tilde{z}_k^{(j-1)}\big) = l, \delta_k(j) = r\big), \end{aligned}$$

where $\ell(\tilde{z}_k^{(j-1)}) = \text{cardinality}\{\tilde{z}_k^{(j-1)}(m) > 0\colon\ 1 \leqslant m \leqslant M\}$ and $\{\tilde{z}_k^{(j-1)}\}$ is under stationarity.

Mutual independence between different queues of $\mathcal{M}^{(i)}$ system is maintained because $\{\delta_k(j)\}$ is assumed to be *with* queue j and is independent of the driving system $\{\tilde{z}_k^{(j-1)}\}$. The proof of the fact that $\mathcal{M}^{(i)}$ upper-bounds $\mathcal{L}^{(i)}$ is similar and, hence, theorem 2.5 goes through in this case.

Extension of results on rates of convergence can be done in a similar manner. The argument for the fact: $\mathcal{L}^{(i)}$ is regenerative, aperiodic with $\mathbb{E}[(\tilde{\tau}(i))^s] < \infty$ implies $\mathcal{L}^{(i+1)}$ is regenerative with $\mathbb{E}[(\tilde{\tau}(i+1))^s] < \infty$, is similar to that in the proof of theorem 3.2.

Under the conditions of the corollary in this section, delay bounds are available for Markov modulated arrival processes. The procedure is almost the same as in the previous section. We skip the details.

Now we provide some examples in support of the above theory.

Example 6.9. Let $M = 3$. Both the arrival and service process are Markov-modulated. The number of states of the modulating Markov chain (for the arrival process) at nodes 1–3 are 2, 3, 2, respectively. The respective transition probability matrices are

$$\begin{bmatrix} 0.95 & 0.05 \\ 0.45 & 0.55 \end{bmatrix}, \qquad \begin{bmatrix} 0.75 & 0.20 & 0.05 \\ 0.25 & 0.20 & 0.55 \\ 0.25 & 0.40 & 0.35 \end{bmatrix}, \qquad \begin{bmatrix} 0.80 & 0.20 \\ 0.30 & 0.70 \end{bmatrix}.$$

These matrices lead to the stationary probability vectors

$$[0.90,\ 0.10], \quad [0.50,\ 0.25,\ 0.25], \quad [0.60,\ 0.40].$$

Now we specify the arrival distribution at each node and each Markov chain state at that node. Only the probabilities for nonzero number of arrivals is specified, the rest is assigned to zero. Let $p(i, j, k)$ denote the probability of k arrivals in a slot in which the Markov chain state is j at node i. Then we specify

$$\begin{array}{llll} p(1,1,1) = 0.2, & p(1,1,0) = 0.8, & p(1,2,1) = 0.1, & p(1,2,3) = 0.033, \\ p(2,1,1) = 0.11, & p(2,2,2) = 0.3, & p(2,3,1) = 0.9, & p(2,3,2) = 0.001, \\ p(3,1,1) = 0.6, & p(3,2,1) = 0.3. & & \end{array}$$

Then we have the arrival rates 0.20, 0.43, 0.48 for the three users. The modulating Markov chain for the service process has two states, and has the transition probability matrix

$$\begin{bmatrix} 0.6 & 0.4 \\ 0.1 & 0.9 \end{bmatrix},$$

with the associated stationary probability vector [0.2, 0.8]. The packet success probabilities are:

$$\begin{aligned} q_1^{(1)} &= 0.9, & q_1^{(2)} &= 1.0, \\ q_2^{(1)} &= 0.8, & q_2^{(2)} &= 0.9, \\ q_3^{(1)} &= 0.7, & q_3^{(2)} &= 0.8. \end{aligned}$$

This leads to the average service probabilities (average with respect to the stationary distribution of the Markov chain) [0.98, 0.88, 0.78]. Now we will vary the arrival distributions keeping the arrival rates the same. The distribution of the modulating Markov chains is kept the same. We consider four cases. In the first case, the arrival distribution is the same as above. For the other three cases, only the additions or changes are indicated below.

The results for the *independent Markov chain case* are given in table 7.

Case 2: $p(1,1,50) = 0.004$, $p(1,1,1) = 0$;

Case 3: $p(1,1,100) = 0.002$, $p(1,1,1) = 0$;

Case 4: $p(2,1,100) = 0.0011$, $p(2,1,1) = 0$.

The results for the *common Markov chain case* (after 1,000,000 iterations) are given in table 6; while for the independent Markov chain case, in table 7.

The above example shows that the r_is depend on the distribution of the arrival processes even in the case of Markov modulated service process. Also, as in the case of *i.i.d.* service process, r_i is close to $\tilde{r}_i$.

Table 6

Case	r_1	r_2	r_3
1	0.78	0.854339	0.905188
2	0.78	0.854559	0.905313
3	0.78	0.854679	0.905291
4	0.78	0.854339	0.905193

Table 7

Case	$r_1 = \tilde{r}_1$	$r_2 = \tilde{r}_2$	r_3	$\tilde{r}_3$
1	0.78	0.854372	0.905477	0.903984
2	0.78	0.854372	0.904896	0.903984
3	0.78	0.854372	0.904728	0.903984
4	0.78	0.854372	0.905584	0.903984

In the next example, all the simulations results have been obtained with $1,000,000$ iterations.

Example 6.10. In this example, we consider seven cases of arrival distributions. The arrival rates are gradually increased for successive cases. The arrival distribution for case 1 is the same as that for the case 1 of previous example. The distribution for the other cases is obtained by changing the entries in the distribution for case 1. Only these changes are indicated below.

Case 2: $p(1,1,1) = 0.3$, $p(1,2,1) = 0.45$, $p(2,1,1) = 0.18$, $p(2,2,2) = 0.335$, $p(2,3,2) = 0.036$, $p(3,1,1) = 0.63$, $p(3,2,1) = 0.33$;

Case 3: $p(1,1,1) = 0.4$, $p(1,2,1) = 0.55$, $p(2,1,1) = 0.28$, $p(2,2,2) = 0.385$, $p(2,3,2) = 0.086$, $p(3,1,1) = 0.73$, $p(3,2,1) = 0.43$;

Table 8

Case	λ_1	λ_2	λ_3	r_1	r_2	r_3
1	0.20	0.43	0.57	0.78	0.854430	0.905204
2	0.40	0.50	0.60	0.78	0.828752	0.870560
3	0.50	0.60	0.70	0.78	0.815878	0.844154
4	0.60	0.70	0.77	0.78	0.803173	0.816796
5	0.70	0.73	0.78	0.78	0.790344	0.798456
6	0.77	0.778	0.78	0.78	0.781415	0.781552
7	0.77	0.78	0.781	0.78	0.781415	0.781335

Table 9

Case	Mean sojourn times			Lower bounds			Upper bounds		
	1	2	3	1	2	3	1	2	3
1	1.536	1.825	1.291	1.305 15	1.487 18	1.052 18	1.760 14	2.514 37	2.084 61
2	1.588	2.213	1.490	1.220 23	1.591 28	1.057 29	1.901 19	2.931 32	2.267 52
3	1.856	3.109	2.140	1.226 33	1.798 42	1.079 49	2.140 15	4.058 30	3.893 81
4	2.448	5.874	5.236	1.254 48	2.119 63	1.105 78	2.670 9	7.892 34	22.884 337
5	4.174	9.866	12.981	1.302 68	2.256 77	1.112 91			
6	27.419	131.253	119.297	1.379 94	2.550 98	1.112 99			
7	28.567	325.260	274.473	1.379 95	2.565 99	1.112 99			

Case 4: $p(1,1,1) = 0.5,\ p(1,2,1) = 0.65,\ p(2,1,1) = 0.38,\ p(2,2,2) = 0.485,$
$p(2,3,2) = 0.086,\ p(3,1,1) = 0.8,\ p(3,2,1) = 0.5;$

Case 5: $p(1,1,1) = 0.6,\ p(1,2,1) = 0.75,\ p(2,1,1) = 0.41,\ p(2,2,2) = 0.515,$
$p(2,3,2) = 0.086,\ p(3,1,1) = 0.81,\ p(3,2,1) = 0.51;$

Case 6: $p(1,1,1) = 0.67,\ p(1,2,1) = 0.82,\ p(2,1,1) = 0.46,\ p(2,2,2) = 0.56,$
$p(2,3,2) = 0.086,\ p(3,1,1) = 0.81,\ p(3,2,1) = 0.51;$

Case 7: $p(1,1,1) = 0.67,\ p(1,2,1) = 0.82,\ p(2,1,1) = 0.462,\ p(2,2,2) = 0.562,$
$p(2,3,2) = 0.086,\ p(3,1,1) = 0.811,\ p(3,2,1) = 0.511.$

The results are given in tables 8–11. For common Markov chain case see tables 8 and 9. Independent Markov chains case is given in tables 10 and 11.

Table 10

Case	$r_1 = \tilde{r}_1$	$r_2 = \tilde{r}_2$	r_3	$\tilde{r}_3$
1	0.78	0.854410	0.905470	0.904025
2	0.78	0.828769	0.870786	0.868378
3	0.78	0.815949	0.844409	0.842353
4	0.78	0.803128	0.817127	0.815907
5	0.78	0.790308	0.798718	0.797875
6	0.78	0.781333	0.781824	0.781760
7	0.78	0.781333	0.781589	0.781504

Table 11

Case	Mean sojourn times			Lower bounds			Upper bounds		
	1	2	3	1	2	3	1	2	3
1	1.525	1.827	1.290	1.303 14	1.488 18	1.051 18	1.771 16	2.497 36	2.068 60
2	1.590	2.196	1.486	1.222 23	1.593 27	1.056 28	1.894 19	2.909 32	2.251 51
3	1.852	3.072	2.136	1.226 33	1.797 41	1.078 49	2.129 14	4.052 31	3.795 77
4	2.432	5.720	5.040	1.251 48	2.113 63	1.105 78	2.684 10	7.834 36	20.139 299
5	4.198	9.495	11.562	1.312 68	2.257 76	1.109 90			
6	26.551	122.232	70.004	1.378 94	2.550 97	1.109 98			
7	27.208	174.343	126.373	1.378 94	2.568 98	1.110 99			

The above example indicates how r_i and $\tilde{r}_i$ fall as the arrival rates gradually increase. On the other hand, the mean delay increases. As in the *i.i.d.* case, the bounds are close to the actual for low arrival rates.

Appendix A. Proof of theorem 2.4

The approach here is a modification of the one presented in [9,24]. Consider a discrete-time slotted system of M queues in which $x_k(i)$ denotes the number of arrivals during slot k at node i, $z_k(i)$ denotes the queue length at the beginning of slot k at node i, and $T_k(i)$ denotes the number of departures during slot k at node i, which could be 1 or 0. The ith queue evolves as

$$z_{k+1}(i) = \big(z_k(i) - T_k(i)\big)^+ + x_k(i),$$

where

$$\mathbb{P}\big(T_k(i) = 1 \mid \ell(z_k) = j\big) = q_j,$$

$\ell(z_k)$ is the cardinality of the set $\{z_k(i) > 0\colon\ 1 \leqslant i \leqslant M\}$. Let $\{x_k(l)\}$ be modulated by an irreducible, aperiodic, finite-state Markov chain $\{\gamma_k(l)\}$ with state space $S_l = \{1, 2, \ldots, L_l\}$. Denote $S = S_1 \times S_2 \times \cdots \times S_M$. Let the modulating chains be mutually independent. Denote $\gamma_k = (\gamma_k(1), \ldots, \gamma_k(M))$. Then $\{(z_k, \gamma_k)\}$ is a Markov chain (denoted by $\mathcal{L}$) taking values in the state space

$$Z_+^M * S = \big\{(z_1, \ldots, z_M; i)\colon\ z_j \geqslant 0 \text{ integers},\ i \in S\big\}.$$

Let $p^k_{(\alpha,i)(\beta,j)}$ be the k-step transition probabilities of $\mathcal{L}$ with $p^1_{(\alpha,i)(\beta,j)} = p_{(\alpha,i)(\beta,j)}$. Also, let

$$N^k(\alpha, i) = \big(N_1^k(\alpha, i), \ldots, N_M^k(\alpha, i)\big)$$

be the vector of mean jumps from the point α in k steps defined as

$$N^k(\alpha, i) = \sum_{(\beta,j)} (\beta - \alpha) p^k_{(\alpha,i)(\beta,j)}.$$

We denote

$$N(\alpha, i) = N^1(\alpha, i).$$

Let

$$R_+^M = \big\{(r_1, \ldots, r_M)\colon\ r_i \geqslant 0 \text{ real}\big\}.$$

Let Λ_M denote the set $\{1, 2, \ldots, M\}$ and $\Lambda = \{i_1, i_2, \ldots, i_k\}$ be any subset of Λ_M. Then corresponding to this Λ, we define a face B^Λ of R_+^M as

$$B^\Lambda = \big\{(r_1, \ldots, r_M)\colon\ r_i > 0,\ i \in \Lambda;\ r_i = 0,\ i \notin \Lambda\big\}.$$

Note that the Markov chain $\mathcal{L}$ satisfies the condition: for any Λ and for all $a \in B^{\Lambda} \cap Z_+^M$,

$$p_{(\alpha,i)(\beta,j)} = p_{(\alpha+a,i)(\beta+a,j)} \quad \forall \alpha \in B^{\Lambda} \cap Z_+^M,\ \forall \beta \in Z_+^M.$$

Thus, we can write

$$p_{(\alpha,i)(\beta,j)} = p(\Lambda, i, j; \beta - \alpha).$$

For any $\Lambda \neq \Lambda_M$, we choose an arbitrary point $a \in B^{\Lambda} \cap Z_+^M$ and draw a plane c^{Λ} of dimension $M - |\Lambda|$ (where $|\Lambda|$ is the cardinality of set Λ) perpendicular to B^{Λ} and containing a. We define the induced Markov chain $\mathcal{L}^{\Lambda}$ with state space $(c^{\Lambda} \cap Z_+^M) * S$, (which will be denoted by c^{Λ} itself for convenience) and transition probabilities

$$p_{\Lambda;(\alpha,i)(\beta,j)} = p_{(\alpha,i)(\beta,j)} + \sum_{\gamma \neq \beta} p_{(\alpha,i)(\gamma,j)} \quad \forall \alpha, \beta \in c^{\Lambda},$$

where the summation is performed over all $\gamma \in Z_+^M$ such that the straight line connecting γ and β is perpendicular to c^{Λ}. Note that this construction does not depend on a.

We assume that for any Λ the chain $\mathcal{L}^{\Lambda}$ is irreducible and aperiodic. We call B^{Λ} ergodic, non-ergodic or transient accordingly as $\mathcal{L}^{\Lambda}$ is ergodic, non-ergodic or transient. For an ergodic $\mathcal{L}^{\Lambda}$, let $\pi^{\Lambda}(\gamma, i)$, $\gamma \in c^{\Lambda}$, $i \in S$, be the stationary probability vector. Introduce the vector $v^{\Lambda} = (v_1^{\Lambda}, \ldots, v_M^{\Lambda})$ by setting

$$v_i^{\Lambda} = \begin{cases} 0, & i \notin \Lambda, \\ \sum_{\gamma \in c^{\Lambda}, j \in S} \pi^{\Lambda}(\gamma, j) N_i(\gamma, j), & i \in \Lambda. \end{cases}$$

Intuitively, imagine that a Markov chain $\mathcal{L}$ starts from a point which is close to B^{Λ}, but sufficiently far from all other faces $B^{\Lambda'}$ with $\Lambda \not\subseteq \Lambda'$ (see [24]). After some time (sufficiently long, but less than the minimal distance from the above mentioned $B^{\Lambda'}$) the stationary regime in the induced chain will be established. In this regime, one can ask about the mean drift along B^{Λ}: it is defined exactly by v^{Λ}. We call Λ_M ergodic by definition and put

$$v^{\Lambda_M} \equiv N(\alpha), \quad \alpha \in B^{\Lambda_M} \cap Z_+^M.$$

We will use the following proposition in proving the theorem.

Proposition A.1. If for some ergodic face Λ, all components of v^{Λ} are positive, then the Markov chain $\mathcal{L}$ is transient.

Proof. Fix an ergodic face Λ and let ψ_t be the corresponding induced Markov chain. Introduce the mutually independent random variables $\eta(t, x, j)$, enumerated by $x \in c^{\Lambda}$,

$j \in S,\ t = 0, 1, 2, \ldots,$ and defined as follows: they take values in $Z^k,\ k = |\Lambda|$, so that

$$\mathbb{P}\big(\eta(t, x, j) = y\big) = \sum_{z \in c^{\Lambda}, l \in S} p_{(\alpha,j)(\beta,l)},$$

where $a, y \in B^{\Lambda}$, $\alpha = (a, x)$, $\beta = (a + y, z)$. Consider the following process with values in $Z^k, k = |\Lambda|$:

$$S_n = S_0 + \sum \eta(t, \psi_t),$$

where $S_0 \in B^{\Lambda} \cap Z_+^M$. This process is a random functional over the induced chain. Since ψ_t is stationary, $\eta(t, \psi_t)$ is also stationary. So, by ergodic theorem, (assume $\psi_0 \equiv (0, 1)$) we have

$$\frac{1}{n} \sum_{t=0}^{n-1} \eta(t, \psi_t) \to v^{\Lambda} \quad \text{a.s.}$$

Then there exist sufficiently small $\varepsilon > 0$ and $n_0 > 0, \delta > 0$ such that for any component $S_{n,i}$ of S_n for $i \in \Lambda$

$$S_0 + \big(v_i^{\Lambda} - \varepsilon\big)n < S_{n,i} < S_0 + \big(v_i^{\Lambda} + \varepsilon\big)n$$

for all $n > n_0$ with probability not less than δ. Now take $S_{0,i}$ sufficiently large in each component. In particular, $S_{0,i} > n_0$ would suffice. Then there exists a set A of trajectories of $\{S_n\}$ with $S_{n,i} > 0\ \forall n, i$ and $S_{n,i} \to \infty,\ \forall i \in \Lambda$. Moreover, this set A has positive probability.

Coming back to the original Markov chain on $Z_+^M * S$, start it from a point having coordinates $S_{0,i}$ for $i \in \Lambda$, zero otherwise, with the Markov chain in state 1. Now we take the set A' of all trajectories $\omega_n,\ n = 0, 1, \ldots,$ of this Markov chain, so that the projection of A' onto B^{Λ} coincides with A. Noting that A and A' have the same probability, the transience is proved. □

Now we state the proof of theorem 2.4. Note that

$$v_j^{\{i+1,i+2,\ldots,M\}} = \lambda_j - r_{i+1}, \quad j \geqslant i + 1,\ 1 \leqslant i \leqslant M - 1.$$

Hence, if $\lambda_{i+1} > r_{i+1}$ for some $i = 0, \ldots, M-1$, then $v_j^{\{i+1,i+2,\ldots,M\}} > 0,\ \forall j \geqslant i+1$. This implies transience from the previous proposition.

The above approach can be extended to prove theorems 6.4 and 6.8, in which it is assumed that the channel assumes a state with the state changes modulated by a Markov chain. For a common Markov chain case (theorem 6.4), we have

$$\mathbb{P}\big(T_k(i)\big) = 1 \mid \ell(z_k) = j,\ \delta_k = l\big) = q_j^{(l)},$$

where $\ell(z_k)$ is the cardinality of the set $\{z_k(i) > 0\colon\ 1 \leqslant i \leqslant M\}$, and $\{\delta_k\}$ is an irreducible, aperiodic Markov chain taking values in a finite state space. In this case, $\{(z_k, \gamma_k, \delta_k)\}$ is a Markov chain. The drifts v_i^{Λ} can be defined in the same way for this system, with the averaging now performed over the states of $\{\delta_k\}$ as well. Theorem 6.4 goes through in this case. The proof of theorem 6.8 is also similar. We drop the details.

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