Source-channel Coding for Gaussian Sources Over a Gaussian Multiple Access Channel

R Rajesh and Vinod Sharma
Dept. of Electrical Communication Engg.
Indian Institute of Science
Bangalore, India
Email: rajesh@pal.ece.iisc.ernet.in,vinod@ece.ernet.in

Abstract— **We consider the problem of distributed joint source-channel coding of correlated Gaussian sources over a Gaussian Multiple Access Channel (MAC). There may be side information at the encoders and/or at the decoder. First we specialize a general result in [16] to obtain sufficient conditions for reliable transmission over a Gaussian MAC. This system does not satisfy the source channel separation. Thus, next we study and compare three joint source channel coding schemes available in literature.**

Keywords: Gaussian MAC, Correlated Gaussian sources, Distributed joint source-channel coding.

I. INTRODUCTION AND SURVEY

Sensor networks are used in a wide variety of applications, the most common being the spatio-temporal monitoring of a random field [2] and the detection of change in its statistics [20]. Sensor networks are characterized by inexpensive sensing nodes with limited battery power and storage and hence limited computing and communication capabilities [2]. These sensor nodes need to transmit their observations to a fusion center which uses this data to estimate the sensed random field. Since transmission is very energy intensive, it is important to minimize it.

The sensor nodes transmit their observations to the fusion center (or a cluster head) usually over a Multiple Access Channel (MAC) ([3], [20]). Often the received symbol is a super-position of the transmitted symbols corrupted by Additive White Gaussian Noise (AWGN). This then is the well known Gaussian MAC (GMAC). This channel is interesting from a practical as well as a theoretical perspective. Also the sensor nodes can be modeled as discrete or continuous sources. For continuous sources, Gaussian distribution is particularly useful. This for example can happen if the sensor nodes are sampling a Gaussian random field. In the application of detection of change, it is often the detection of change in the mean of the sensor observation with the sensor observation noise being Gaussian ([20]). Thus, in this paper we focus on transmission of Gaussian sources over a GMAC.

In the following we survey the related literature. Cover, El Gamal and Salehi [4] provided sufficient conditions for transmitting losslessly discrete correlated observations over a discrete MAC. They also show that unlike for independent sources, the source-channel separation does not hold. However the single letter characterization of the capacity region obtained in [4] is only a sufficient condition. Indeed Duek [6] proved that the conditions given in [4] are only sufficient and may not be necessary. A related paper on this problem is [1]. The results of [4] have been extended in [19] to the system with side information and distortion.

The distributed Gaussian source coding problem is discussed in [12], [21]. The exact rate region for two users is provided in [21].

A Gaussian sensor network in distributed and collaborative setting is studied in [10]. The authors show that it is better to compress the local estimates than to compress the raw data. In [14] it is shown that feed back increases the capacity of a GMAC. GMAC under received power constraints is studied in [7] and it is shown that the source-channel separation holds in this case. A GMAC with lossy transmission of discrete sources and side information is studied in [16]. The GMAC with correlated jointly Gaussian input is studied in [15] and the edge capacities obtained are similar to the expressions in [14]. In [11] one necessary and two sufficient conditions for transmitting a jointly Gaussian source over a GMAC are provided. The authors prove that the (uncoded) amplify and forward scheme is optimal below a certain SNR. This SNR is a function of the correlation between the sources. In [8] the authors discuss a joint source channel coding scheme over a MAC and show the scaling behavior for the Gaussian channel. Actually their problem is closer to the Gaussian CEO problem [13]. The new element is that the coded observations are transmitted through a multiple access channel. The scaling laws for the problem without side information are discussed in [9] and it is shown that the standard practice of separating source coding from channel coding may require exponential increase in bandwidth, as the number of sensors increases. A lower bound on best achievable distortion as a function of the number of sensors, total transmit power, the number of degrees of the underlying process and the spatio-temporal communication bandwidth is obtained.

This paper makes the following contributions. From our general results in [16] we obtain explicit conditions for transmission of correlated Gaussian sources with given distortion over a GMAC. Also we compare the two schemes in [11] with a separation based scheme. We explicitly show that the amplify and forward scheme in [11] is not optimal at high SNR. Furthermore, for this scheme, it may not be optimal to use all the power. Also, from our general results we recover a sufficient condition for transmission provided in [11] without proof. Furthermore we provide the results with side information also.

The paper is organized as follows. Sufficient conditions for transmission of continuous correlated sources over a continuous MAC are given in Section II. The transmission of Gaussian sources on a Gaussian MAC is discussed in Section III. Different joint source-channel coding schemes for transmission are studied and their asymptotic performances are compared. In Section IV optimal power allocation to minimize the sum of the distortions for the three schemes is obtained. Section V summarizes the results.

II. CORRELATED SOURCES OVER A MAC

In this section we state a general result obtained for transmission of continuous alphabet sources over a continuous alphabet MAC with side information. This result was obtained in [16] and is a generalization of the result presented in [19]. In the next section we specialize it to Gaussian sources and a Gaussian MAC. We present the result for two sources. However it is available for more than two sources in [16]. Source i generates an iid sequence $\{U_{in}, n \geq 1\}, i = 1, 2$. The outputs U_{1n} and U_{2n} may have dependence. The MAC is assumed to be memoryless. Its transition function is $f(y|x_1, x_2)$ where x_1 and x_2 are the channel inputs and y is the channel output. At time n, the encoder i also has iid side information $\{Z_{ik}, k \leq n\}, i = 1, 2$ available to it. The decoder at the receiver has side information $\{Z_{1n}, n \geq 1\}, \{Z_{2n}, n \geq 1\}$ and $\{Z_n, n \geq 1\}$ available causally.

We are interested in knowing if for given average transmit power constraints P_1, P_2, at the encoders 1 and 2 respectively, it is possible to transmit $\{U_{1n}, U_{2n}, n \geq 1\}$ over the channel with given distortions (D_1, D_2). For simplicity we assume that (U_1, U_2, Z_1, Z_2, Z) has a density $f(u_1, u_2, z_1, z_2, z)$. Also distortion functions are assumed to be bounded. The following theorem holds ([16]).

Theorem 1: A source (U_1, U_2) can be communicated in a distributed fashion over a continuous alphabet MAC $f(y|x_1, x_2)$ with distortions (D_1, D_2) if we can find random variables W_1, W_2, X_1, X_2, such that

1. $$\begin{aligned} f(u_1, u_2, z_1, z_2, z, w_1, w_2, x_1, x_2, y) = \\ f(u_1, u_2, z_1, z_2, z) f(w_1|u_1, z_1) f(w_2|u_2, z_2) \\ f(x_1|w_1) f(x_2|w_2) f(y|x_1, x_2). \end{aligned}$$

2. For any given $\epsilon > 0$ there exists an $N > 0$ and for any $n \geq N$, a decoder $f_D : \mathcal{W}_1^n \times \mathcal{W}_2^n \times \mathcal{Z}_1^n \times \mathcal{Z}_2^n \times \mathcal{Z}^n \rightarrow (\hat{\mathcal{U}}_1^n \times \hat{\mathcal{U}}_2^n)$ such that $\frac{1}{n} E\left[\sum_i^n d(U_{ij}, \hat{U}_{ij})\right] \leq D_i + \epsilon,\ i = 1, 2$ and the constraints

$$\begin{aligned} I(U_1; W_1|W_2, Z_1, Z_2, Z) &< I(X_1; Y|X_2, W_2, Z_1, Z_2, Z), \\ I(U_2; W_2|W_1, Z_1, Z_2, Z) &< I(X_2; Y|X_1, W_1, Z_1, Z_2, Z), \\ I(U_1, U_2; W_1, W_2|Z_1, Z_2, Z) &< I(X_1, X_2; Y|Z_1, Z_2, Z). \end{aligned} \tag{1}$$

are satisfied along with the prescribed transmit power constraints $E[X_i^2] \leq P_i,\ i = 1, 2$. $\mathcal{U}_i,\ \mathcal{Z}_i,\ \mathcal{Z},\ \mathcal{W}_i,\ \mathcal{X}_i,\ \mathcal{Y},\ \hat{\mathcal{U}}_i$ are the sets in which $U_i,\ Z_i,\ Z,\ W_i,\ X_i,\ Y,\ \hat{U}_i$ take values. ■

We specialize this result to the Gaussian sources and GMAC in the next section.The main problem in applying the result in the above theorem in specific examples is to obtain a good coding scheme. Thus we will also study the performance of three coding schemes. In this paper we will mostly consider the system without side information Z_1, Z_2, Z. Then the inequalities in (1) become

$$\begin{aligned} I(U_1; W_1|W_2) &< I(X_1; Y|X_2, W_2), \\ I(U_2; W_2|W_1) &< I(X_2; Y|X_1, W_1), \\ I(U_1, U_2; W_1, W_2) &< I(X_1, X_2; Y). \end{aligned} \tag{2}$$

It has been shown in [16] that under our conditions $I(X_1; Y|X_2, W_2) \leq I(X_1; Y|X_2)$ and $I(X_2; Y|X_1, W_1) \leq I(X_2; Y|X_1)$.

III. GAUSSIAN SOURCES OVER GAUSSIAN MAC

In a Gaussian MAC the channel output Y_n at time n is given by $Y_n = X_{1n} + X_{2n} + N_n$ where X_{1n} and X_{2n} are the channel inputs at time n and N_n is a Gaussian random variable independent of X_{1n} and X_{2n}, with $E[N_n] = 0$ and $var(N_n) = \sigma^2$. We will also assume that (U_{1n}, U_{2n}) is jointly Gaussian with mean zero, variances $\sigma_i^2,\ i = 1, 2$ and correlation ρ. The distortion measure will be Mean Square Error (MSE). This does violate the boundedness of distortion measure assumed in Theorem 1. However, if we modify MSE as $E[min(\overline{d}, (U_i - \hat{U}_i)^2)]$ with $\overline{d}$ very large (w.r.t σ_i^2) the error induced in the distortion measure will be insignificant.

It is shown in [16] that $I(X_1; Y|X_2),\ I(X_2; Y|X_1)$ and $I(X_1, X_2; Y)$ are maximized by zero mean Gaussian random variables (X_1, X_2) with variances P_1 and P_2. Also if correlation between X_1 and X_2 is $\tilde{\rho}$ then the conditions in (2) become (using the relaxation mentioned below (2))

$$\begin{aligned} I(U_1; W_1|W_2) &< 0.5 \log\left[1 + \frac{P_1(1-\tilde{\rho}^2)}{\sigma_N{}^2}\right], \\ I(U_2; W_2|W_1) &< 0.5 \log\left[1 + \frac{P_2(1-\tilde{\rho}^2)}{\sigma_N{}^2}\right], \\ I(U_1, U_2; W_1, W_2) &< 0.5 \log\left[1 + \frac{P_1 + P_2 + 2\tilde{\rho}\sqrt{P_1 P_2}}{\sigma_N{}^2}\right]. \end{aligned} \tag{3}$$

In the rest of the paper we consider three specific coding schemes to obtain W_1, W_2, X_1, X_2 where (W_1, W_2) satisfy the distortion constraints in the Theorem and (X_1, X_2) are jointly Gaussian with an appropriate $\tilde{\rho}$ such that (3) is satisfied. These coding schemes have been available before. Our purpose is to compare their performance.

A. Amplify and forward scheme

In the amplify and forward (AF) scheme the channel codes X_i are just scaled source symbols U_i. Since (U_1, U_2) are themselves jointly Gaussian, (X_1, X_2) will be jointly Gaussian and retain the dependence of inputs (U_1, U_2). The

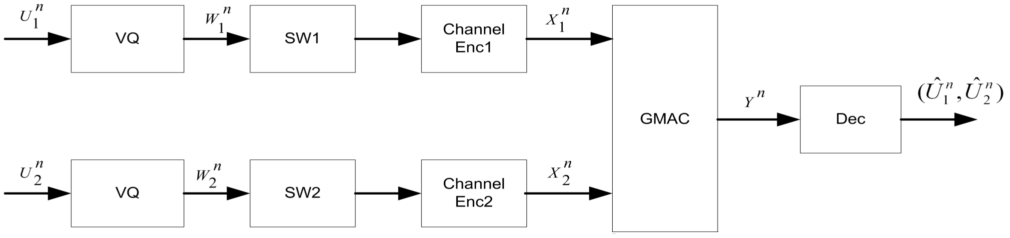

Fig. 1. separation based scheme

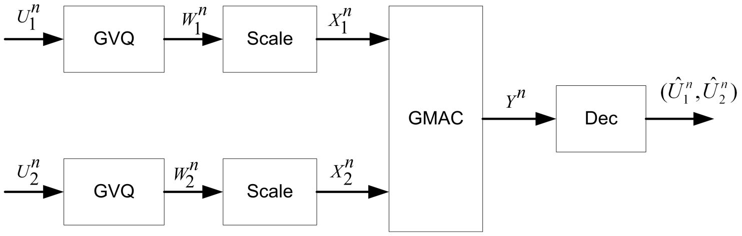

Fig. 2. Joint source channel coding

scaling is done to ensure $E[{X_i}^2] = P_i, i = 1, 2$. For a single user case this coding is optimal [5]. At the decoder inputs U_1 and U_2 are directly estimated from Y as $\hat{U}_i = E[U_i|Y], i = 1, 2$. Because U_i and Y are jointly Gaussian this estimate is linear and also satisfies the Minimum Mean Square Error (MMSE) and Maximum Likelihood (ML) criteria. Thus the MMSE distortion for this encoding-decoding scheme is

$$D_1 = \frac{{\sigma_1}^2 \left[P_2(1-\rho^2) + {\sigma_N}^2\right]}{P_1 + P_2 + 2\rho\sqrt{P_1 P_2} + {\sigma_N}^2}, \tag{4}$$

$$D_2 = \frac{{\sigma_2}^2 \left[P_1(1-\rho^2) + {\sigma_N}^2\right]}{P_1 + P_2 + 2\rho\sqrt{P_1 P_2} + {\sigma_N}^2}. \tag{5}$$

Since encoding and decoding require minimum processing and delay in this scheme, if it satisfies the distortion bounds D_i, it should be the scheme to implement. This scheme has been studied in [11] and found to be optimal below a certain SNR for two-user symmetric case $(P_1 = P_2, \sigma_1 = \sigma_2, D_1 = D_2)$. However unlike for single user case, in this case user 1 acts as interference for user 2 (and vice versa). Thus one should not expect this scheme to be optimal under high SNR case. We will show that this is indeed true.

B. Separation based approach

In separation based (SB) approach (Fig. 1) the jointly Gaussian sources are vector quantized to W_1^n and W_2^n. The quantized outputs are Slepian-Wolf encoded [18]. This produces code words, which are (asymptotically) independent. These independent code words are encoded to capacity achieving Gaussian channel codes (X_1^n, X_2^n) with correlation $\tilde{\rho} = 0$. This is a very natural scheme and has been considered by various authors ([4], [5], [17]).

Since source-channel separation does not hold for this system, this scheme is not expected to be optimal. However we will see later that this scheme may perform almost as well as the best joint source-channel coding scheme available. Of course, from practical point of view, this scheme is preferable to a joint source-channel coding scheme with comparable performance.

C. Joint source-channel coding

The optimal scheme for this system is a joint source-channel coding (JSC) scheme (Fig. 2) which provides (X_1^n, X_2^n) to be jointly Gaussian with high correlation. Therefore, here we draw on the results in [11] and [21]. From [21] we can obtain a vector quantizer providing W_1^n and W_2^n jointly Gaussian and satisfying the distortion constraints. Alternatively we can use the coding in [11] to obtain (U_1, U_2, W_1, W_2) (approximately) jointly Gaussian with covariance matrix given in (6). In (6) R_1 and R_2 are the rates of the vector quantizers to quantize U_1 and U_2 and $\tilde{\rho} = \rho\sqrt{(1-2^{-2R_1})(1-2^{-2R_2})}$. In this coding (Fig. 2) since X_i is obtained from W_i by scaling, $\tilde{\rho}$ is also the correlation between X_1 and X_2.

We obtain the (R_1, R_2) needed in (6) from (3). From

$I(U_1; W_1|W_2) = H(W_1|W_2) - H(W_1|W_2, U_1)$,

and the fact that the Markov chain condition $W_1 \leftrightarrow U_1 \leftrightarrow U_2 \leftrightarrow W_2$ holds,

$H(W_1|W_2, U_1) = H(W_1|U_1)$ and

$I(U_1; W_1|W_2) = 0.5 \log\left[(1-\tilde{\rho}^2)2^{2R_1}\right]$.

Thus from (3) we need

$$R_1 \leq 0.5 \log\left[\frac{P_1}{{\sigma_N}^2} + \frac{1}{(1-\tilde{\rho}^2)}\right]. \tag{7}$$

Similarly, we also need

$$\begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 & \sigma_1^2(1-2^{-2R_1}) & \rho\sigma_1\sigma_2(1-2^{-2R_2}) \\ \rho\sigma_1\sigma_2 & \sigma_2^2 & \rho\sigma_1\sigma_2(1-2^{-2R_2}) & \sigma_2^2(1-2^{-2R_2}) \\ \sigma_1^2(1-2^{-2R_1}) & \rho\sigma_1\sigma_2(1-2^{-2R_1}) & \sigma_1^2(1-2^{-2R_1}) & \frac{\tilde{\rho}\sigma_1\sigma_2}{\rho} \\ \rho\sigma_1\sigma_2(1-2^{-2R_2}) & \sigma_2^2(1-2^{-2R_2}) & \frac{\tilde{\rho}\sigma_1\sigma_2}{\rho} & \sigma_2^2(1-2^{-2R_2}) \end{pmatrix} \tag{6}$$

$$R_2 \le 0.5 \log \left[\frac{P_2}{\sigma_N{}^2} + \frac{1}{(1-\tilde{\rho}^2)} \right], \text{and} \tag{8}$$

$$R_1 + R_2 \le 0.5 \log \left[\frac{\sigma_N{}^2 + P_1 + P_2 + 2\tilde{\rho}\sqrt{P_1 P_2}}{(1-\tilde{\rho}^2)\sigma_N{}^2} \right]. \tag{9}$$

The inequalities (7)-(9) are the same as in [11]. Thus we recover the conditions in [11] from our general result (1). The estimation of U_1, U_2 from W_1, W_2 gives the conditions for existence of auxiliary random variables (W_1, W_2) for distortions D_1 and D_2

$$D_1 \ge var(U_1|W_1, W_2) = \frac{\sigma_1{}^2 2^{-2R_1}\left[1-\rho^2\left(1-2^{-2R_2}\right)\right]}{(1-\tilde{\rho}^2)} \tag{10}$$

$$D_2 \ge var(U_2|W_1, W_2) = \frac{\sigma_2{}^2 2^{-2R_2}\left[1-\rho^2\left(1-2^{-2R_1}\right)\right]}{(1-\tilde{\rho}^2)}. \tag{11}$$

The lower bound is achieved for $\hat{U}_i = E[U_i|W_1, W_2], i = 1, 2$. The minimum distortion is obtained when $\tilde{\rho}$ is such that the sum rate is met with equality in (9). For the symmetric case at the minimum distortion, $R_1 = R_2 = R$.

D. Asymptotic performance of the three schemes

We compare the performance of the three schemes. Even though the AF and the JSC schemes have been studied in [11], their performance have not been compared. For simplicity we consider the symmetric case: $P_1 = P_2 = P$, $\sigma_1 = \sigma_2 = \sigma$, $D_1 = D_2 = D$. We will denote the SNR $\frac{P}{\sigma_N{}^2}$ by S.

1) Amplify and forward scheme: Consider the AF scheme. From (4)

$$D(S) = \frac{\sigma^2\left[S\left(1-\rho^2\right)+1\right]}{2S\left(1+\rho\right)+1}. \tag{12}$$

Thus $D(S)$ decreases to $\frac{\sigma^2(1-\rho)}{2}$ strictly monotonically at rate $O(1)$ as $S \to \infty$. Hence we find that even as the channel SNR increases the distortion in the AF scheme does not go to zero. As we commented before, this happens because as $S \to \infty$ the interference from the other user also increases.

Next we consider the performance at low SNR. Then

$$\lim_{S \to 0}\left|\frac{D(S)-\sigma^2}{S}\right| = \lim_{S \to 0}\frac{\sigma^2(1+\rho)^2}{2S\left(1+\rho\right)+1} = \sigma^2(1+\rho)^2 \tag{13}$$

Thus, $D(S) \to \sigma^2$ at rate $O(S)$ as $S \to 0$.

2) Separation based scheme: Consider the SB scheme at High SNR. From [21] if each source is encoded with rate R then it can be decoded at the decoder with distortion

$$D^2 = 2^{-4R}(1-\rho^2) + \rho^2 2^{-8R}. \tag{14}$$

At high SNR, from the capacity result for independent inputs, we have $R < 0.25 \log S$ ([5]). Then from (14) we obtain

$$D \ge \sqrt{\frac{\sigma^4(1-\rho^2)}{S} + \frac{\sigma^4\rho^2}{S^2}} \tag{15}$$

and this lower bound is achievable. As $S \to \infty$, this lower bound approaches zero at rate $O(\sqrt{S})$. Thus this scheme outperforms AF at high SNR.

At low SNR, $R \approx \frac{S}{2}$ and hence from (14)

$$D \ge \rho^2\sigma^4 2^{-4S} + \sigma^2(1-\rho^2)2^{-2S}. \tag{16}$$

Thus $D \to \sigma^2$ at rate $O(S^2)$ as $S \to 0$ at high ρ and at rate $O(S)$ at small ρ. Therefore we expect that at low SNR, at high ρ this scheme will be worse than AF but at low ρ it will be comparable.

3) Joint source-channel coding: Consider the JSC scheme. In the high SNR region we assume that $\tilde{\rho} = \rho$ since $R = R_1 = R_2$ are sufficiently large. Then from (8) $R \approx 0.25 \log\left[\frac{2S}{1-\rho}\right]$ and the distortion can be approximated by

$$D \approx \sigma^2\sqrt{\frac{1-\rho}{2S}}. \tag{17}$$

Therefore, $D \to 0$ as $S \to \infty$ at rate $O(\sqrt{S})$. This rate of convergence is same as for the separation based scheme. However, the RHS in (15) is greater than that of (17) and at low ρ the two are close. Thus at high SNR JSC always outperforms SB but the improvement is small for low ρ.

At low SNR $R \approx \dfrac{S(1+\tilde{\rho})}{2} - \dfrac{\log(1-\tilde{\rho}^2)}{4}$ and evaluating D from (10) we get

$$D = \frac{\sigma^2 2^{-\overline{S}}\left(1-\rho^2(1-\sqrt{1-\tilde{\rho}^2}2^{-\overline{S}})\right)}{\sqrt{1-\tilde{\rho}^2}} \tag{18}$$

where $\overline{S} = S(1+\tilde{\rho})$. Therefore $D \to \sigma^2$ as $S \to 0$ at rate $O(S^2)$ at high ρ and at rate $O(S)$ at low ρ. These rates are same as that for SB. In fact, dividing the expression for D at low SNR for SB by that for JSC, we can show that the two distortions tend to σ^2 at the same rate for all ρ .

The necessary conditions (NC) to be able to transmit on the GMAC with distortion (D, D) for the symmetric case are ([11])

$$D \geq \begin{cases} \frac{\sigma^2\left[S\left(1-\rho^2\right)+1\right]}{2S(1+\rho)+1}, & \text{for } S \leq \frac{\rho}{1-\rho^2}, \\ \sigma^2\sqrt{\frac{(1-\rho^2)}{2S(1+\rho)+1}}, & \text{for } S > \frac{\rho}{1-\rho^2}. \end{cases} \quad (19)$$

The above three schemes along with (19) are compared below using exact computations. Figs 3-5 show the distortion as a function of SNR for unit variance jointly Gaussian sources with correlations ρ = 0.1, 0.5 and 0.75.

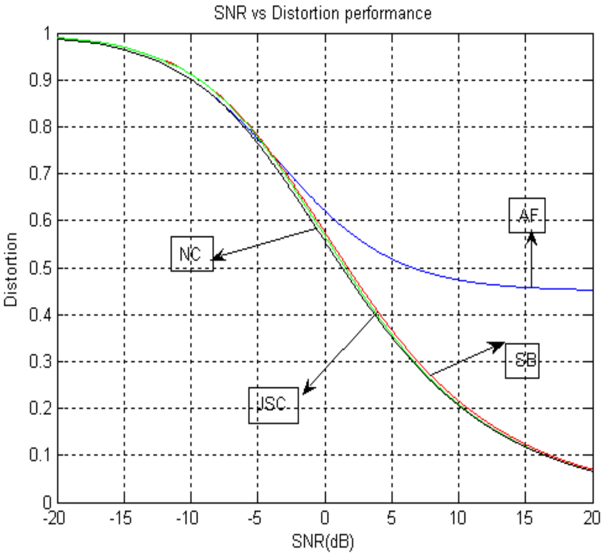

Fig. 3. SNR vs distortion performance for $\rho = .1$

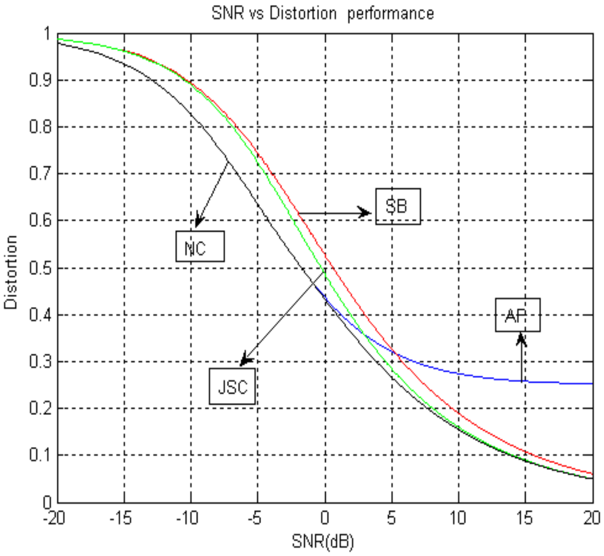

Fig. 4. SNR vs distortion performance for $\rho = .5$

From these plots we confirm our theoretical conclusions provided above. In particular we obtain that:
(i) AF is close to necessary conditions and hence optimal (as shown in [11]) at low SNR. The other two schemes perform worse at low SNR.
(ii) SB and JSC schemes perform better than AF at high SNRs.
(iii) JSC scheme performs better than SB scheme in general.
(iv) Performance of SB and JSC are close for low ρ and for high ρ at low SNR.

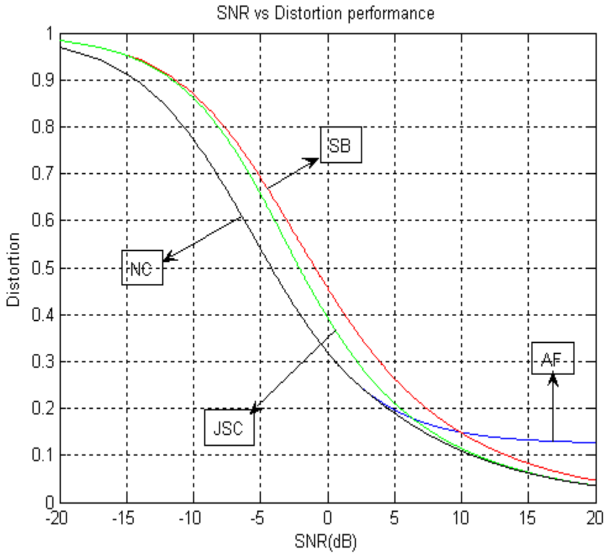

Fig. 5. SNR vs distortion performance for $\rho = .75$

4) Performance with side information: The asymptotic results can be extended to the case with side-information only at the decoder. Let the side information random variables be $Z_1 = aU_2 + n_1, Z_2 = bU_1 + n_2$, where n_1 and n_2 are zero mean unit variance Gaussian random variables independent of U_1 and U_2 and independent of each other. The constants a and b can be considered as side channel SNR.

Each of the above schemes can be directly extended to this case by using the same encoders as above (i.e., without side information (Z_1, Z_2)). But at the decoder use side information (Z_1, Z_2) also.

For AF scheme, X_i is a scaled version of U_i and at the decoder $\hat{U}_i = E[U_i|Y, Z_1, Z_2], i = 1, 2$. Then the distortion D is given in (20).

At high SNR the minimum distortion in (20) approaches $\frac{\sigma^2(1-\rho)}{2+\sigma^2(1-\rho)(a^2+b^2)}$ which is strictly positive, as in no-side information case. However this limit is lower than for the no-side information case.

For JSC scheme, we obtain W_i^n and X_i^n from U_i^n as before. At the decoder we obtain $(\hat{W_1^n}, \hat{W_2^n})$ that is jointly typical with Y^n. Then we obtain $(\hat{U_1^n}, \hat{U_2^n})$ from $(\hat{W_1^n}, \hat{W_2^n}, Z_1^n, Z_2^n)$ as $\hat{U_1^n} = E[U_i^n|\hat{W_1^n}, \hat{W_2^n}, Z_1^n, Z_2^n]$ which under our assumptions becomes $E[U_{ik}|\hat{W}_{1k}, \hat{W}_{2k}, Z_{1k}, Z_{2k}]$, $i = 1, 2;\ k = 1, 2 \ldots n$.

The comparison of the three schemes with side channel SNR $a = b = 1$ and $\rho = 0.5$ is shown in Fig 6. As before AF is the best at low SNR but performs worse than the other schemes at high SNR. JSC scheme performs better than the

$$D \geq \frac{\sigma^2\left[(1-\rho^2)(P+a\sigma^2\sigma_N^2)+\sigma_N^2\right]}{\sigma_N^2+2P(1+\rho)+(1-\rho^2)\left[P\sigma^2(a^2+b^2)\right]+\sigma^2\sigma_N^2\left[a^2+b^2+a^2b^2(1-\rho^2)\right]} \tag{20}$$

SB scheme at high SNR. Comparing these results with those in Fig 4 provide the value of the side information.

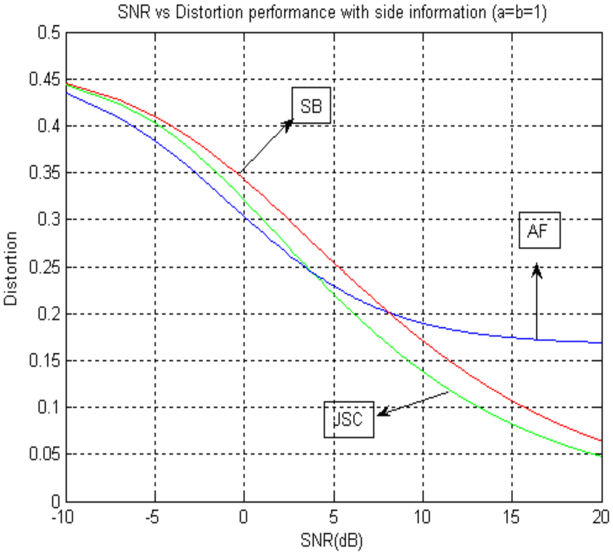

Fig. 6. SNR vs distortion with side information for $\rho = .5$

We have been able to improve the performance of SB and JSC schemes in this setup by exploiting the side information Z_1 and Z_2 at the encoders also, even though this side information is directly not available at the encoders. Our work on this is still in progress. This along with the results on side information at the encoders will be reported later.

IV. OPTIMAL POWER ALLOCATION

Till now we studied the performance of the three coding schemes for given average powers P_1 and P_2. The question arises if indeed it is optimal to expand the whole of P_1 and P_2 or one could do better by using lower average powers. In this section, using the performance obtained in previous sections we address this issue.

Without loss of generality we take the sources with unit variance (when they have different variances, by scaling we can reduce the problem to equal, unit variance case). Also, let $\sigma_N^2 = 1$. The objective is to find for the three schemes, the minimum distortion for the given power constraints P_1 and P_2 and also the optimal powers.

A. *Amplify and forward scheme*

For the AF, we find powers a^* and b^* that minimize

$$w_1D_1 + w_2D_2 = w_1\frac{b(1-\rho^2)+1}{a+b+2\sqrt{ab}\rho+1} + w_2\frac{a(1-\rho^2)+1}{a+b+2\sqrt{ab}\rho+1} \tag{21}$$

subject to

$$a \leq P_1 \text{ and } b \leq P_2$$

where w_1, w_2 are given positive weights.

Unlike for a single user case, the optimal powers a^* and b^* may not be equal to P_1 and P_2 respectively. Therefore, we solve this optimization problem using the Kuhn-Tucker first order conditions. In the following we also study the qualitative behavior of the cost function to get more insight into the problem.

From Section III(D) for symmetric case we know that D strictly decreases as a function of P. Hence then for $w_1 = w_2$, $a^* = b^* = P$. In general, if we use powers a and b, then

$$D_1 + D_2 > \frac{(a+b)(1-\rho^2)}{a+b+2\sqrt{ab}\rho} \geq \frac{1-\rho^2}{1+\rho} = 1-\rho.$$

The second inequality follows because $a+b \geq 2\sqrt{ab}$ and equality is achieved only if $a = b$. The first inequality will tend to equality if a or $b \rightarrow \infty$. Thus for large P_1 and P_2, $a^* \approx b^*$. The minimum D_1+D_2 approaches the lower bound $1-\rho$ as P_1 and $P_2 \rightarrow \infty$. However if we keep a fixed and increase b then the first lower bound starts increasing after sometimes. Thus if P_1 is fixed and P_2 is increased, b^* will not keep increasing.

From (21) we find that D_1 is monotonically decreasing with a. Next fix a and vary b. If a and b are large compared to 1 and if $a \ll b$ then $D_1 \approx 1-\rho^2$. Thus D_1 is insensitive to b if $1 \ll a \ll b$. This can be interpreted as follows. At high b the noise can be neglected and when $1 \ll a \ll b$ at the decoder U_2 is available accurately and Y will be close to U_2. Thus $E[U_1|U_2]$ will be close $E[U_1|Y]$ and the mean square error $E\left[(U_1-\hat{U}_1)^2\right] \approx 1-\rho^2$. By symmetry, same follows for D_2 when $1 \ll b \ll a$.

Next consider the case where $1 \ll b \ll a$. Then $D_1 \approx \dfrac{b(1-\rho^2)}{a}$. Here D_1 is sensitive to b and increases with b irrespective of ρ. This may be because the interference effect of U_2 dominates as b increases.

We illustrate by a few examples.

Example 1. Consider $\rho = 0.5$ and $w_1 = w_2 = 1$. For various values of P_1, P_2 the optimal solution is given in Table I for the symmetric case. We see that for the symmetric case, the minimum distortion is achieved when the power constraints are equal. Also as power increases the distortion tends to $1-\rho = 0.5$.

Next consider the case where P_1 is fixed and P_2 is varying. The optimal values of a^*, b^* and D are provided in Table II. In this case b^* can be less than P_2. This happens because when a is fixed, $D_1 + D_2$, as a function

TABLE I

POWER ALLOCATION FOR THE SYMMETRIC CASE

P_1	P_2	a^*	b^*	$D_{min} = D_1 + D_2$
1	1	1	1	0.8750
5	5	5	5	0.5937
10	10	10	10	0.5484
50	50	50	50	0.5099
100	100	100	100	0.5050

of b, is strictly decreasing in the interval $[0, c)$ and strictly increasing in (c, ∞) where

$$c = \left[\frac{(1+\rho^2)+\sqrt{(1+\rho^2)^2+4a\rho^2(1-\rho^2)[a(1-\rho^2)+2]}}{2\sqrt{a}\rho(1-\rho^2)} \right]^2$$

Then $b = c$ is the unique global minimum of $D_1 + D_2$ for a given a. For $\rho = .5$ and $a = 10$, $c = 17.01$ as can be observed from Table II. Table II also shows that for $P_2 > P_1$, $a^* = P_1$ and the optimum b^* is obtained at c for $a^* = P_1$.

TABLE II

POWER ALLOCATION FOR THE ASYMMETRIC CASE

P_1	P_2	a^*	b^*	D_{min}
10	1	10	1	0.6760
10	5	10	5	0.5743
10	20	10	17.01	0.5422
10	50	10	17.01	0.5422

When $w_1 \neq w_2$, even when $P_1 = P_2$, the optimal a, b may be less than P_i. For $\rho = 0.5, w_1 = 0.4$ and $w_2 = 0.6$, the optimal point is obtained in Table III.

TABLE III

POWER ALLOCATION WITH DIFFERENT WEIGHTS

P_1	P_2	a^*	b^*	$D_{min} = w_1 D_1 + w_2 D_2$
1	1	1	1	0.4375
5	5	4.69	5	0.2968
10	10	5.69	10	0.2707
100	100	29.9	100	0.2395

B. Separation based scheme

For the SB scheme, X_1, X_2 are independent zero mean Gaussian random variables. The capacity region for this case monotonically increases with P_1, P_2. Thus, the distortion $D_1 + D_2$ will be minimized at the powers $(a^*, b^*) = (P_1, P_2)$.

C. Joint source-channel coding

The optimization problem for the JSC scheme is:

Minimize

$$\frac{\sigma_1^2 2^{-2R_1}[1-\rho^2(1-2^{-2R_2})]}{1-\tilde{\rho}^2} + \frac{\sigma_2^2 2^{-2R_2}[1-\rho^2(1-2^{-2R_1})]}{1-\tilde{\rho}^2} \qquad (22)$$

Subject to

$$\begin{aligned} R_1 &\leq 0.5 \log \left(a + \frac{1}{1-\tilde{\rho}^2} \right), \\ R_2 &\leq 0.5 \log \left(b + \frac{1}{1-\tilde{\rho}^2} \right), \\ R_1 + R_2 &\leq 0.5 \log \left(\frac{1}{1-\tilde{\rho}^2} + \frac{a+b+2\sqrt{ab}\tilde{\rho}}{1-\tilde{\rho}^2} \right), \\ & a \leq P_1, \ b \leq P_2. \end{aligned}$$

From the constraints we find that R_1, R_2 and $R_1 + R_2$ are monotonically increasing in a, b and $\tilde{\rho}$. Hence these are maximized at $(a, b) = (P_1, P_2)$. Also the objective function is monotonically decreasing in R_1 and R_2 (the first derivative of the objective function is negative for all non-negative(R_1, R_2)). Hence the minimum distortion is achieved at powers (P_1, P_2).

Let $\rho = 0.5$ and $\sigma_1^2 = \sigma_2^2 = \sigma_N^2 = 1$ with varying P_1 and P_2. The results are shown in Table IV and it is observed that $D_1 + D_2$ is minimized at the powers $(a, b) = (P_1, P_2)$ for JSC. Comparing Table IV with the results in Tables I and II we see that for $P_1 = P_2 = 1$ (0dB SNR) AF provides less distortion while for $P_1 = P_2 = 10$ (10dB SNR) JSC provides less distortion.

TABLE IV

POWER ALLOCATION FOR JSC SCHEME

P_1	P_2	a^*	b^*	$\tilde{\rho}$	D_{min}
1	1	1	1	0.2398	0.9720
5	5	5	5	0.3798	0.4553
10	10	10	10	0.4160	0.3217
10	20	10	20	0.4314	0.2645
10	50	10	50	0.4504	0.1927

V. CONCLUSIONS

We have specialized a general result discussed in [16] to the Gaussian sources and the Gaussian MAC. We analyze three joint source-channel coding schemes available in the literature and compare their distortion performance. We prove that the amplify and forward scheme is sub-optimal at high SNR's. Cases with side information at the decoder are addressed. We also provide optimum power allocation policies for the three schemes.

REFERENCES

[1] R. Ahlswede and T. Han, "On source coding with side information via a multiple access channel and related problems in information theory," *IEEE Trans. Inform. Theory*, vol. IT-29, no. 3, pp. 396–411, 1983.

[2] I. F. Akylidiz, W. Su, Y. Sankarasubramaniam, and E. Cayirici, "A survey on sensor networks," *IEEE Communications Magazine*, pp. 1–13, august 2002.

[3] S. J. Baek, G. Veciana, and X. Su, "Minimizing energy consumption in large-scale sensor networks through distributed data compression and hierarchical aggregation," *IEEE JSAC*, vol. 22, no. 6, pp. 1130–1140, Aug 2004.

[4] T. M. Cover, A. E. Gamel, and M. Salehi, "Multiple access channels with arbitrarily correlated sources," *IEEE Trans. Inform. Theory*, vol. IT -26, pp. 648–657, 1980.

[5] T. M. Cover and J. A. Thomas, *Elements of Information theory*. Wiley Series in Telecommunication, N.Y., 2004.

[6] G. Dueck, “A note on the multiple access channel with correlated sources,” *IEEE Trans. Inform. Theory*, vol. IT -27, no. 2, pp. 232–235, 1981.

[7] M. Gastpar, “Multiple access channels under received-power constraints,” *Proc. IEEE Inform. Theory Workshop*, pp. 452–457, 2004.

[8] M. Gastpar and M. Vetterli, “Source-channel communication in sensor networks,” *Proc. IPSN’03*, pp. 162–177, 2003.

[9] ——, “Power spatio-temporal bandwidth and distortion in large sensor networks,” *IEEE JSAC*, vol. 23, no. 4, pp. 745–754, 2005.

[10] P. Ishwar, R. Puri, K. Ramchandran, and S. S. Pradhan, “On rate constrained distributed estimation in unreliable sensor networks,” *IEEE JSAC*, pp. 765–775, 2005.

[11] A. Lapidoth and S. Tinguely, “Sending a bi- variate Gaussian source over a Gaussian MAC,” *IEEE International Symp. on Inform. Theory 06*, 2006.

[12] Y. Oohama, “Gaussian multiterminal source coding,” *IEEE Trans. Inform. Theory*, vol. IT -43, no. 6, pp. 1912–1923, 1997.

[13] ——, “The rate distortion function for quadratic Gaussian CEO problem,” *IEEE Trans. Inform. Theory*, vol. IT -44, pp. 1057–1070, 1998.

[14] L. H. Ozarow, “The capacity of the white Gaussian multiple access channel with feedback,” *IEEE Trans. Inform. Theory*, vol. IT -30, pp. 623 – 629, 1984.

[15] S. S. Pradhan, S. Choi, and K. Ramchandran, “A graph based framework for transmission of correlated sources over multiple access channels,” *Arxiv*, 2005. [Online]. Available: http://arxiv.org/pdf/cs.IT/0608006

[16] R. Rajesh, V. K. Varsheneya, and V. Sharma, “Distributed joint source-channel coding on a multiple access channel with side information,” *Submitted*.

[17] S. Ray, M. Medard, M. Effros, and R. Kotter, “On separation for multiple access channels,” *Proc. IEEE Inform. Theory Workshop*, 2006.

[18] D. Slepian and J. K. Wolf, “Noiseless coding of correlated information sources,” *IEEE Trans. Inform. Theory*, vol. IT -19, pp. 471–480, 1973.

[19] V. K. Varsheneya and V. Sharma, “Distributed coding for multiple access communication with side information,” *Proc. IEEE Wireless Communication and Networking Conference (WCNC)*, April 2006.

[20] V. V. Veeravalli, “Decentralized quickest change detection,” *IEEE Trans. Inform. Theory*, vol. IT-47, pp. 1657–65, 2001.

[21] A. B. Wagner, S. Tavildar, and P. Viswanath, “The rate region of the quadratic Gaussian two terminal source coding problem,” *Arxiv*, 2005. [Online]. Available: http://arxiv.org/pdf/cs.IT/0510095