A Tight Rate Bound and a Matching Construction for Locally Recoverable Codes with Sequential Recovery From Any Number of Multiple Erasures

S. B. Balaji, Ganesh R. Kini and P. Vijay Kumar, Fellow, IEEE
Department of Electrical Communication Engineering, Indian Institute of Science, Bangalore.
Email: balaji.profess@gmail.com, kiniganesh94@gmail.com, pvk1729@gmail.com

Abstract

An \([n,k]\) code \(C\) is said to be locally recoverable in the presence of a single erasure, and with locality parameter \(r\), if each of the \(n\) code symbols of \(C\) can be recovered by accessing at most \(r\) other code symbols. An \([n,k]\) code is said to be a locally recoverable code with sequential recovery from \(t\) erasures, if for any set of \(s \leq t\) erasures, there is an \(s\)-step sequential recovery process, in which at each step, a single erased symbol is recovered by accessing at most \(r\) other code symbols. This is equivalent to the requirement that for any set of \(s \leq t\) erasures, the dual code contain a codeword whose support contains the coordinates of precisely one of the \(s\) erased symbols.

In this paper, we derive a tight upper bound on the rate of such a code, for any value \(r \geq 3\), of the locality parameter. The rate bound also proves an earlier conjecture due to Song, Cai and Yuen. While the bound is valid irrespective of the field over which the code is defined, we provide here a matching construction of binary codes that are rate-optimal, i.e., binary codes achieving the rate bound for any \(t\) and any \(r \geq 3\).

Index Terms

Distributed storage, locally recoverable codes, codes with locality, locally repairable codes, sequential repair, multiple erasures, rate bound, proof of conjecture.

I. INTRODUCTION

An \([n,k]\) code \(C\) is said to have locality \(r\) if each of the \(n\) code symbols of \(C\) can be recovered by accessing at most \(r\) other code symbols. Equivalently, there exist \(n\) codewords \(h_1,\ldots,h_n\), not necessarily distinct, in the dual code \(\overline{C}\), such that \(i \in \text{supp}(h_i)\) and \(|\text{supp}(h_i)| \leq r + 1\) for \(1 \leq i \leq n\) where \(\text{supp}(h_i)\) denotes the support of the codeword \(h_i\).

a) Codes with Sequential Recovery: An \([n,k]\) code \(C\) over a field \(F_q\) is defined as a code with sequential recovery from \(t\) erasures and with locality-parameter \(r\), if for any set of \(s \leq t\) erased symbols \(\{c_{\sigma_1},\ldots,c_{\sigma_s}\}\), there exists a codeword \(h\) in the dual code \(\overline{C}\) of \(C\) of Hamming weight \(\leq r + 1\), such that \(\text{supp}(h) \cap \{\sigma_1,\ldots,\sigma_s\} = 1\). We will formally refer to this class of codes as \((n,k,r,t)_{\text{seq}}\) codes. When the parameters \((n,k,r,t)\) are clear from the context, we will simply refer to a code in this class as a code with sequential recovery.

A. Background

In [2] Gopalan et al. introduced the concept of codes with locality (see also [3], [4]), where an erased code symbol is recovered by accessing a small subset of other code symbols. The size of this subset denoted by \(r\), is typically much smaller than the dimension of the code, making the repair process more efficient when compared with MDS codes. The focus of [2] was local recovery from single erasure (see also [2], [3], [4], [5]).

The sequential approach to recovery from erasures, introduced by Prakash et al. [8] is one of several approaches to locally recover from multiple erasures. Codes employing this approach have been shown to be better in terms of rate and minimum distance (see [3], [9], [10], [11], [12]). Local recovery in the presence of two erasures is considered in [8] (see also [10]) where a tight rate bound for two erasure case and an optimal construction is provided. Codes with sequential recovery from three erasures can be found discussed in [10], [11], [13]. A bound on rate of an \((n,k,r,3)_{\text{seq}}\) code was derived in [13]. A rate bound for \(t = 4\) appears in [12].

There are several approaches to local recovery from multiple erasures:

1) Stronger Local Codes: Here, every code symbol is contained in a local code of length at most \(r + t\) and minimum distance at least \(t + 1\), see (12). The recovery process is sequential in any order. For more details see [6], [15].

2) Codes with Availability: For every code symbol \(c_i\) in \(C\), there exist \(t\) codewords \(h^{i}_1,\ldots,h^{i}_t\) in the dual of the code, each of Hamming weight \(\leq r + 1\), such that \(\text{supp}(h^{i}_g) \cap \text{supp}(h^{i}_j) = \{i\}, \forall 1 \leq g \neq j \leq t\); recovery can be carried out in parallel. For more details see [16], [12], [7], [18].
3) **Codes with Selectable Recovery:** Here, given any set of \( t \) erasures, every erased symbol has a parity check that involves that symbol and no other erased symbol; with these codes, one is free to choose the order in which to recover the erased symbols; recovery in parallel may or may not be possible, depending upon the construction. For more details see [19].

4) **Codes with Sequential Recovery:** Already defined and is the theme of this paper. The class of Sequential recovery codes under consideration here, is a larger class of codes that contains all the three above-mentioned classes of codes as depicted in figure [1]. For this reason, codes with sequential recovery can potentially achieve higher rate and have larger minimum distance.

A fourth class of codes, not covered here, are termed as codes with cooperative local recovery, see [9].

![Fig. 1: Figure showing that codes with sequential recovery contain certain other important classes of codes.](image)

**B. Our Contributions**

In this paper, we derive an upper bound on the rate of a code having locality-parameter \( r \) with sequential recovery from \( t \) erasures, for any \( r \geq 3 \) and any \( t \). While the bound is valid irrespective of the field over which the code is defined, we provide here a matching construction of binary codes that are rate-optimal, i.e., binary codes achieving the rate bound for any \( t \) and any \( r \geq 3 \). These results are also shown to prove a conjecture on code rate due to Song, Cai and Yeun [10].

**II. UPPER BOUND ON RATE OF AN \((n, k, r, t)_{seq}\) CODE**

In this section we provide an upper bound on the rate of an \((n, k, r, t)_{seq}\) code for \( r \geq 3 \). The cases of \( t \) even and \( t \) odd are considered in two separate subsections.

**A. Upper Bound on Rate of an \((n, k, r, t)_{seq}\) Code for \( t \) Even:**

In this subsection we provide an upper bound on the rate of an \((n, k, r, t)_{seq}\) code for \( t \) even and \( r \geq 3 \).

**Theorem 1.** Let \( \mathcal{C} \) be an \((n, k, r, t)_{seq}\) code over a field \( \mathbb{F}_q \). Let \( t \) be a positive even integer and \( r \geq 3 \). Then

\[
\frac{k}{n} \leq \frac{r^2}{r^2 + 2 \sum_{i=0}^{r-1} r^i}.
\]

**Proof.** We begin by setting \( \mathcal{B}_0 = \text{span}(\xi \in \mathcal{C}^\perp : w_H(\xi) \leq r + 1) \). Let \( m \) be the dimension of \( \mathcal{B}_0 \). Let \( \xi_1, \ldots, \xi_m \) be a basis of \( \mathcal{B}_0 \) such that \( w_H(\xi_i) \leq r + 1 \).

Let

\[
H_1 = \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_m \end{bmatrix}.
\]

It follows that \( H_1 \) is a parity check matrix of an \((n, k, r, t)_{seq}\) code as its row space contains all the codewords of Hamming weight at most \( r + 1 \) which are contained in \( \mathcal{C}^\perp \). Also,

\[
\frac{k}{n} \leq 1 - \frac{m}{n}.
\]

The idea behind the next few arguments in the proof is the following. The codes with largest rate will tend to have a larger value of \( n \) for fixed \( m \). On the other hand, the Hamming weight of the matrix \( H_1 \) (i.e., the number of non-zero entries in the matrix) is bounded above by \( m(r + 1) \). It follows that to make \( n \) large, one would like the columns of \( H_1 \) to have as small a weight as possible. It is therefore quite natural to start building \( H_1 \) by picking as many columns of weight 1 as possible, then columns of weight 2 and so on. As one proceeds by following this approach, it turns out that the matrix \( H_1 \) is forced to have a certain sparse, block-diagonal, staircase form and an understanding of this structure is used to derive the upper bound on
code rate. We now proceed to derive an upper bound on $1 - \frac{m}{n}$. Without loss of generality the matrix $H_1$, after permutation of rows and columns can be written in the following form

$$H_1 = \begin{bmatrix}
D_0 & A_1 & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & D_1 & A_2 & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & D_2 & A_3 & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & A_{\frac{j}{2}-2} & 0 & 0 \\
0 & 0 & 0 & 0 & \ldots & D_{\frac{j}{2}-2} & A_{\frac{j}{2}-1} & 0 \\
0 & 0 & 0 & 0 & \ldots & 0 & D_{\frac{j}{2}-1} & C \\
0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0
\end{bmatrix},$$

where

1) the rows of $H_1$ are labeled $1, 2, \ldots, m$ and columns are labeled $1, 2, \ldots, n$,
2) $A_i$ is a $\rho_i \times a_i$ matrix for $1 \leq i \leq \frac{j}{2} - 1$, $D_i$ is a $\rho_i \times a_i$ matrix for $0 \leq i \leq \frac{j}{2} - 1$ for some $\{\rho_i\}, \{a_i\}$,
3) $D_0$ is a matrix with each column having weight (Hamming weight) 1 and each row having weight at least 1,
4) $\{A_i\}, \{D_i\}, \{B_i = \left[\frac{A_i}{D_i}\right]\}$ are such that for $1 \leq i \leq \frac{j}{2} - 1$ each column of $B_i$ has weight 2, each column of $A_i$ has weight at least 1 and each row of $D_i$ has weight at least 1 and each column of $D_i$ has weight at most 1,
5) $C$ is a matrix with each column having weight 2, $D$ is a matrix which exactly contains all the columns of $H_1$ which have weight $\geq 3$,
6) If $J$ is the first index such that $A_J, D_J$ ($J = 0$ if $D_0$ is an empty matrix) are empty matrices ($0 \times L, L \times 0, 0 \times 0$ matrix) then we set $A_i, D_i$ to be empty matrices for all $J \leq i \leq \frac{j}{2} - 1$ and set $a_i = 0, \rho_i = 0, J \leq i \leq \frac{j}{2} - 1$ and place all of the 2-weight columns apart from the 2-weight columns corresponding to $B_1$ to $B_{J-1}$ in $C$. Let the number of columns in $C$ be $a_{\frac{j}{2}}$. If $C$ has no columns then we set $a_{\frac{j}{2}} = 0$.

The entire rate bound derivation is correct and all the bounds will hold with $a_i = 0, \rho_i = 0, J \leq i \leq \frac{j}{2} - 1$. If $l < J$ is the first index such that $D_l$ is an empty matrix with $A_l$ a non empty matrix then the proof of the following claim [1] (since in the proof of claim [1] we prove the claim for $A_l$ first and then proceed to $D_l$ and since $D_0, A_j, D_j$ must be non empty $\forall j < l$) will imply that $A_l$ has column weight 1 which will imply $D_l$ cannot be empty. Hence the case $A_l$, a non empty matrix and $D_l$, an empty matrix cannot occur. Although we have to prove the following claim [1] for $A_i, D_i, 1 \leq i \leq J - 1, D_0$, we assume all $D_0, A_i, D_i, 1 \leq i \leq \frac{j}{2} - 1$ to be non-empty matrices and prove the claim. Since the proof is by induction, the induction can be made to stop at $J = 1$ (induction starts at 0) and the proof is unaffected by it.

**Claim 1.** For $1 \leq i \leq \frac{j}{2} - 1$, $A_i$ is a matrix with each column having weight 1 and for $0 \leq j \leq \frac{j}{2} - 1$, $D_j$ is a matrix with each row and each column having weight 1.

**Proof.** We use the fact that $d_{\text{min}}(C) \geq t + 1$ to prove the above claim. Proof is given in Appendix [A].

By Claim [1] after permutation of columns of $H_1$ in (2) within the columns labeled by the set $\{\sum_{l=0}^{j-1} a_l + 1, \ldots, \sum_{l=0}^{j-1} a_l + a_j\}$ for $0 \leq j \leq \frac{j}{2} - 1$, the matrix $D_j, 0 \leq j \leq \frac{j}{2} - 1$ can be assumed to be a diagonal matrix with non-zero entries along the diagonal and hence $\rho_i = a_i$. By counting the row weights and column weights of $A_i, 1 \leq i \leq \frac{j}{2} - 1$ we obtain (Note that if $A_j$ is an empty matrix then also the following inequality is true as we would have set $a_j = 0$):

$$\begin{align*}
\rho_{i-1}r &\geq a_i, \\
a_{i-1}r &\geq a_i.
\end{align*}$$

(3)

For some $p \geq 0$,

$$\begin{align*}
\frac{j}{2} - 1 &\sum_{i=0}^{j-1} a_i + p = m.
\end{align*}$$

(4)

Let the number of columns in matrix $C$ be $a_{\frac{j}{2}}$. Counting the row weights and column weights of the matrix $C$, we obtain (Note that if $C$ is an empty matrix then also the following inequality is true as we would have set $a_{\frac{j}{2}} = 0$):

$$\begin{align*}
2a_{\frac{j}{2}} &\leq (a_{\frac{j}{2}-1} + p)(r + 1) - a_{\frac{j}{2}-1}.
\end{align*}$$

(5)
Let us prove the following inequality by induction for \(0< t\):

\[
J = \frac{t}{2}. 
\]

Substituting (4) in (5):

\[
2a_{\frac{t}{2}} \leq (m - \sum_{i=0}^{\frac{t}{2}-2} a_i)(r+1) - (m - \sum_{i=0}^{\frac{t}{2}-2} a_i - p),
\]

\[
2a_{\frac{t}{2}} \leq (m - \sum_{i=0}^{\frac{t}{2}-2} a_i)r + p. 
\]

(6)

Counting the row weights and column weights of \(H_1\), we get:

\[
m(r+1) \geq a_0 + 2\left(\sum_{i=1}^{\frac{t}{2}} a_i\right) + 3(n - \sum_{i=0}^{\frac{t}{2}} a_i),
\]

\[
m(r+1) \geq 3n - 2a_0 - \left(\sum_{i=1}^{\frac{t}{2}} a_i\right). 
\]

(7)

Our basic inequalities are (3), (5), (4), (7). We manipulate these 4 inequalities to derive the bound on rate.

Substituting (4) in (7):

\[
m(r+1) \geq 3n - a_0 - a_{\frac{t}{2}} - (m - p),
\]

\[
m(r+2) \geq 3n + p - a_0 - a_{\frac{t}{2}}. 
\]

(8)

Substituting (8) in (8), we get:

\[
m(r+2) \geq 3n + p - a_0 - \left(\frac{(m - \sum_{i=0}^{\frac{t}{2}-2} a_i)r + p}{2}\right),
\]

\[
m(r+2) + r \geq 3n + p - a_0 + \left(\frac{\sum_{i=0}^{\frac{t}{2}-2} a_i}{2}\right)r. \]

(9)

From (4), for any \(0 \leq j \leq \frac{t}{2} - 2:

\[
a_{\frac{t}{2} - j} = m - \sum_{i=0}^{\frac{t}{2} - j - 1} a_i - \sum_{i=\frac{t}{2} - j + 1}^{\frac{t}{2} - 1} a_i - p. 
\]

(10)

Substituting (3) for \(\frac{t}{2} - 2 - j + 1 \leq i \leq \frac{t}{2} - 1\) in (10), we get:

\[
a_{\frac{t}{2} - 2 - j} \geq m - \sum_{i=0}^{\frac{t}{2} - 2 - j - 1} a_i - \sum_{i=\frac{t}{2} - 2 - j + 1}^{\frac{t}{2} - 2 - j} a_{\frac{t}{2} - j}r^{i-(\frac{t}{2} - 2 - j)} - p,
\]

\[
a_{\frac{t}{2} - 2 - j} \geq m - \sum_{i=0}^{\frac{t}{2} - 2 - j - 1} a_i - \sum_{i=1}^{j+1} a_{\frac{t}{2} - j}r^i - p,
\]

\[
a_{\frac{t}{2} - 2 - j} \geq \frac{m - \sum_{i=0}^{\frac{t}{2} - 2 - j - 1} a_i - \sum_{i=1}^{j+1} a_{\frac{t}{2} - j}r^i - p}{1 + \sum_{i=1}^{j+1} r^i}. 
\]

(11)

Let

\[
\delta_0 = \frac{r}{2}, 
\]

\[
\delta_{j+1} = \delta_j - \frac{\delta_j}{1 + \sum_{i=1}^{j+1} r^i}. 
\]

(12)

(13)

Let us prove the following inequality by induction for \(0 \leq J \leq \frac{t}{2} - 2,\)

\[
m(r+2 + \delta_J) \geq 3n + p \left(\frac{1}{2} + \delta_j - \frac{r}{2}\right) - a_0 + \left(\sum_{i=0}^{\frac{t}{2} - 2 - j} a_i\right)\delta_J. 
\]

(14)

(14) is true for \(J = 0\) by (9). Hence (14) is proved for \(t = 4\) and the range of \(J\) is vacuous for \(t = 2\). Hence assume \(t > 4.\)
Hence let us assume \((14)\) is true for \(J\) such that \(\frac{t}{2} - 3 \geq J \geq 0\) and prove it for \(J + 1\). Substituting \((11)\) for \(j = J\) in \((14)\), we get:

\[
m(r + 2 + \delta_J) \geq 3n + p \left( \frac{1}{2} + \delta_J - \frac{r}{2} \right) - a_0 + \left( \sum_{i=0}^{\frac{t}{2} - J - 1} a_i \right) \delta_J + \left( \frac{m - \sum_{i=0}^{\frac{t}{2} - J - 1} a_i - p}{1 + \sum_{i=1}^{\frac{t}{2} - J - 1} r^i} \right) \delta_J,
\]

\[
m(r + 2 + \delta_J - \frac{\delta_J}{1 + \sum_{i=1}^{\frac{t}{2} - J - 1} r^i}) \geq 3n + p \left( \frac{1}{2} + \delta_J - \frac{\delta_J}{1 + \sum_{i=1}^{\frac{t}{2} - J - 1} r^i} - \frac{r}{2} \right) - a_0 + \left( \sum_{i=0}^{\frac{t}{2} - J - 1} a_i \right) \left( \delta_J - \frac{\delta_J}{1 + \sum_{i=1}^{\frac{t}{2} - J - 1} r^i} \right). \tag{15}
\]

Substituting \((13)\) in \((15)\), we obtain

\[
m(r + 2 + \delta_{J+1}) \geq 3n + p \left( \frac{1}{2} + \delta_{J+1} - \frac{r}{2} \right) - a_0 + \left( \sum_{i=0}^{\frac{t}{2} - J - 1} a_i \right) \delta_{J+1}. \tag{16}
\]

Hence \((14)\) is proved for any \(0 \leq J \leq \frac{t}{2} - 2\) for \(t \geq 4\). Hence writing \((14)\) for \(J = \frac{t}{2} - 2\) for \(t \geq 4\), we obtain:

\[
m(r + 2 + \delta_{\frac{t}{2} - 2}) \geq 3n + p \left( \frac{1}{2} + \delta_{\frac{t}{2} - 2} - \frac{r}{2} \right) - a_0 + (a_0)\delta_{\frac{t}{2} - 2},
\]

\[
m(r + 2 + \delta_{\frac{t}{2} - 2}) \geq 3n + p \left( \frac{1}{2} + \delta_{\frac{t}{2} - 2} - \frac{r}{2} \right) + a_0(\delta_{\frac{t}{2} - 2} - 1). \tag{17}
\]

It can be seen that \(\delta_J\) for \(r \geq 2\) has a product form as:

\[
\delta_J = \frac{r}{2} \left( \frac{a_{r+1} - a_i}{a^i - 1} \right). \tag{18}
\]

Hence for \(r \geq 3, t \geq 4:\n
\[
\delta_{\frac{t}{2} - 2} = \frac{r}{2} \left( \frac{a_{\frac{t}{2} - 1} - a_{\frac{t}{2} - 2}}{a_{\frac{t}{2} - 2} - 1} \right) \geq 1.
\]

Hence we can substitute \((11)\) for \(j = \frac{t}{2} - 2\) in \((17)\):

\[
m(r + 2 + \delta_{\frac{t}{2} - 2}) \geq 3n + p \left( \frac{1}{2} + \delta_{\frac{t}{2} - 2} - \frac{r}{2} \right) + \left( \frac{m - p}{1 + \sum_{i=1}^{\frac{t}{2} - 1} r^i} \right) (\delta_{\frac{t}{2} - 2} - 1),
\]

\[
m(r + 2 + \delta_{\frac{t}{2} - 2} - \frac{\delta_{\frac{t}{2} - 2}}{1 + \sum_{i=1}^{\frac{t}{2} - 1} r^i} + \frac{1}{1 + \sum_{i=1}^{\frac{t}{2} - 1} r^i}) \geq 3n + p \left( \frac{1}{2} + \delta_{\frac{t}{2} - 2} - \frac{\delta_{\frac{t}{2} - 2}}{1 + \sum_{i=1}^{\frac{t}{2} - 1} r^i} + \frac{1}{1 + \sum_{i=1}^{\frac{t}{2} - 1} r^i} - \frac{r}{2} \right) \tag{19}
\]

Substituting \((13)\) in \((19)\), we obtain:

\[
m \left( r + 2 + \delta_{\frac{t}{2} - 1} + \frac{1}{1 + \sum_{i=1}^{\frac{t}{2} - 1} r^i} \right) \geq 3n + p \left( \frac{1}{2} + \delta_{\frac{t}{2} - 1} + \frac{1}{1 + \sum_{i=1}^{\frac{t}{2} - 1} r^i} - \frac{r}{2} \right). \tag{20}
\]

Using \((18)\), we obtain:

\[
\left( \frac{1}{2} + \delta_{\frac{t}{2} - 1} + \frac{1}{1 + \sum_{i=1}^{\frac{t}{2} - 1} r^i} - \frac{r}{2} \right) \geq 0.
\]

Hence \((20)\) implies:

\[
m \left( r + 2 + \delta_{\frac{t}{2} - 1} + \frac{1}{1 + \sum_{i=1}^{\frac{t}{2} - 1} r^i} \right) \geq 3n. \tag{21}
\]

\((21)\) after some algebraic manipulations gives the required bound on \(1 - \frac{a_0}{m}\) and hence on \(\frac{k}{n}\) as stated in the theorem. Note that although the derivation is valid for \(r \geq 3, t \geq 4\), the final bound given in the theorem is correct and tight for \(t = 2\). The bound for \(t = 2\) can be derived specifically by substituting \(a_0 \leq m\) in \((9)\) and noting that \(p \geq 0\). \(\square\)

An alternative proof for Theorem \([1]\) is given below by using linear programming:
**Proof.** The inequalities (3), (5) and (7) are linear inequalities and are written in matrix form as:

\[ A\mathbf{x} \geq \mathbf{b} \]

where

\[
A = \begin{bmatrix}
  r & -1 & 0 & \ldots & 0 & 0 & 0 \\
  0 & r & -1 & \ldots & 0 & 0 & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
  0 & 0 & 0 & \ldots & r & -1 & 0 \\
  0 & 0 & 0 & \ldots & 0 & r & -2 (r + 1) \\
  (r + 3) & (r + 2) & (r + 2) & \ldots & (r + 2) & (r + 2) & 1 (r + 1)
\end{bmatrix}
\]

which is a \((\frac{t}{2} + 1) \times (\frac{t}{2} + 2)\) matrix and

\[
\mathbf{x} = \begin{bmatrix} a_0 & a_1 & \ldots & a_{\frac{t}{2}} & p \end{bmatrix}^T, \quad \mathbf{b} = \begin{bmatrix} 0 & 0 & \ldots & 0 & 3n \end{bmatrix}^T
\]

where \(\mathbf{x}\) is a \((\frac{t}{2} + 2) \times 1\) matrix and \(\mathbf{b}\) is a \((\frac{t}{2} + 1) \times 1\) matrix. The problem of finding an upper bound on rate of the code now becomes one of minimizing \(m = \mathbf{c}^T\mathbf{x}\), which is a linear objective function where \(\mathbf{c} = \begin{bmatrix} 1 & 1 & \ldots & 1 & 0 & 1 \end{bmatrix}^T\) is a \((\frac{t}{2} + 2) \times 1\) matrix. Also by definition of \(\mathbf{x}\), \(\mathbf{x} \geq 0\). This is now in a standard form of a linear program formulation as:

\[
\text{minimize } \mathbf{c}^T\mathbf{x} \\
\text{s.t. } A\mathbf{x} \geq \mathbf{b}, \quad \mathbf{x} \geq 0
\]

The dual problem of the above is

\[
\text{maximize } \mathbf{b}^T\lambda \\
\text{s.t. } A^T\lambda \leq \mathbf{c}, \lambda \geq 0
\]

We will solve an the dual problem by writing it in standard minimize \(-\mathbf{b}^T\lambda\) form and using the simplex method. Let us introduce slack variables \(s_1, \ldots, s_{\frac{t}{2}+2}\) and re-write the constraints as

\[
B\mathbf{y} = \mathbf{c}, \quad \mathbf{y} \geq 0,
\]

where

\[
B = \begin{bmatrix}
  r & 0 & 0 & \ldots & 0 & (r + 3) & 0 & 1 & 0 & 0 & \ldots & 0 & 0 \\
  -1 & r & 0 & \ldots & 0 & (r + 2) & 0 & 0 & 1 & 0 & \ldots & 0 & 0 \\
  0 & -1 & r & \ldots & 0 & (r + 2) & 0 & 0 & 0 & 1 & \ldots & 0 & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  0 & 0 & \ldots & -1 & r & (r + 2) & 0 & 0 & 0 & 0 & \ldots & 1 & 0 \\
  0 & 0 & 0 & \ldots & -2 & 1 & 0 & 0 & 0 & 0 & \ldots & 0 & 1 \\
  0 & 0 & 0 & \ldots & (r + 1) & (r + 1) & 1 & 0 & 0 & 0 & \ldots & 0 & 0
\end{bmatrix},
\]

and

\[
\mathbf{y} = \begin{bmatrix} \lambda_1 & \lambda_2 & \ldots & \lambda_{\frac{t}{2}+1} & s_1 & \ldots & s_{\frac{t}{2}+2} \end{bmatrix}^T.
\]

With this, the objective function now is \(\mathbf{d}^T\mathbf{y}\), where \(\mathbf{d} = \begin{bmatrix} \mathbf{c}^T & 0 & 0 & \ldots & 0 \end{bmatrix}^T\). We pick the variables \(\lambda_1, \ldots, \lambda_{\frac{t}{2}+1}, s_1\) as “basic variables” and the rest, called “non-basic variables” will be set to 0. A set of basic variables is chosen such that the columns of \(B\) corresponding to those variables is a full-rank square matrix. The system of equations is now in the following form:

\[
\begin{bmatrix} B_{BV} & B_{NBV} \end{bmatrix} \begin{bmatrix} \lambda_{BV} \\ \mathbf{0} \end{bmatrix} = \mathbf{c}
\]

Therefore we will equivalently solve

\[
B_{BV}\lambda_{BV} = \mathbf{c}
\]
The above system of equations can be solved in closed form to get the following:

\[
\begin{align*}
\lambda_{\frac{j}{2}+1} &= \frac{2}{3} \sum_{i=0}^{\frac{j}{2}-1} r^i, \\
s_1 &= \frac{1}{r^2 + 2 \sum_{i=0}^{\frac{j}{2}-1} r^i}, \\
\lambda_{j+1} &= \frac{r^2 - 3r^{\frac{j}{2}-(j+1)} + 2}{3(r-1)(r^2 + 2 \sum_{i=0}^{\frac{j}{2}-1} r^i)}, \text{ for } 0 \leq j \leq \frac{t}{2} - 1
\end{align*}
\]

which are non-negative if \( r \geq 3 \). Hence the chosen basic solution is a basic feasible solution. To check for optimality we check if the “reduced cost coefficients” \( r_i = b_i - z_{\alpha_i} \) are non-negative, for every non-basic variable \( \alpha_i \). We note that for the above made choice of non-basic variables, \( b_i = 0 \). Suppose the non-basic variables are labeled \( \alpha_1, ..., \alpha_M \) and the basic variables are labeled \( \beta_1, ..., \beta_N, z_{\alpha_i} \) is defined as follows:

\[
z_{\alpha_i} = \sum_{\beta = \beta_i}^{\beta_N} b_{\beta} y(\beta, \alpha_i) = b_{\beta_i} y(\beta_{\beta_i} + 1, \alpha_i) = -3n y(\beta_{\beta_i} + 1, \alpha_i)
\]

where \( y(\beta_{\beta_i} + 1, \alpha_i) \) are as shown in the row reduced echelon form of matrix \( B \) below:

\[
B_{rref} = \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 & 0 & y(1, \alpha_1) & y(1, \alpha_2) & \cdots & y(1, \alpha_{M-1}) & y(1, \alpha_M) \\
0 & 1 & 0 & \cdots & 0 & 0 & y(2, \alpha_1) & y(2, \alpha_2) & \cdots & y(2, \alpha_{M-1}) & y(2, \alpha_M) \\
0 & 0 & 1 & \cdots & 0 & 0 & y(3, \alpha_1) & y(3, \alpha_2) & \cdots & y(3, \alpha_{M-1}) & y(3, \alpha_M) \\
& \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & 0 & y(\frac{j}{2} + 1, \alpha_1) & y(\frac{j}{2} + 1, \alpha_2) & \cdots & y(\frac{j}{2} + 1, \alpha_{M-1}) & y(\frac{j}{2} + 1, \alpha_M) \\
0 & 0 & 0 & \cdots & 0 & 1 & y(\frac{j}{2} + 2, \alpha_1) & y(\frac{j}{2} + 2, \alpha_2) & \cdots & y(\frac{j}{2} + 2, \alpha_{M-1}) & y(\frac{j}{2} + 2, \alpha_M) \\
0 & 0 & 0 & \cdots & 0 & 0 & 1 & y(\frac{j}{2} + 3, \alpha_1) & y(\frac{j}{2} + 3, \alpha_2) & \cdots & y(\frac{j}{2} + 3, \alpha_{M-1}) & y(\frac{j}{2} + 3, \alpha_M)
\end{bmatrix}
\]

We observe that in reducing row-\( \frac{j}{2} + 1 \) we add only non-negative linear combinations of the rows above it, entries of which are either 0 or 1. Therefore \( r_{\alpha_i} \geq 0 \) for \( \alpha_i \) all non-basic variables. Hence \( y(\frac{j}{2} + 1, \alpha_1), ..., y(\frac{j}{2} + 1, \alpha_M) \geq 0 \). Hence the chosen basic solution is an “optimal basic feasible” solution.

By the theorem of strong duality the optimal solutions of the primal problem and the dual problem are equal. Therefore the minimum value of \( m \) is 

\[
m_{\frac{j}{2}+1} = \frac{3n}{(r^2 + 2 \sum_{i=0}^{\frac{j}{2}-1} r^i)}.
\]

Hence we get the upper bound on the rate:

\[
\frac{k}{n} \leq 1 - \frac{r^2}{n} \leq \frac{r^2}{r^2 + 2 \sum_{i=0}^{\frac{j}{2}-1} r^i},
\]

We now pick a solution for the primal problem and show that it is feasible and gives the optimal objective function value.

\[
a_i = \frac{2nr^i}{r^2 + 2 \sum_{i=0}^{\frac{j}{2}-1} r^i}, \text{ for } 0 \leq i \leq \frac{t}{2} - 1
\]

\[
a_{\frac{j}{2}} = \frac{nr^2}{r^2 + 2 \sum_{i=0}^{\frac{j}{2}-1} r^i}, \quad p = 0.
\]

It is easy to check that this solution satisfies the first \( \frac{j}{2} \) constraints of the primal problem with equality. It remains to check the following:

\[
(r + 3)a_0 + (r + 2) \sum_{i=1}^{\frac{j}{2}-1} a_i + a_{\frac{j}{2}} \geq 3n.
\]

Upon simplification, it is seen that the above is met with equality. Therefore the chosen solution is a feasible solution. It is also easy to check that the solution gives the optimal value of the objective function. Hence it is an optimal feasible solution. We thus conclude that a code having the above chosen values could have the optimal rate.

\[\Box\]

B. Upper Bound on Rate of an \((n, k, r, t)\) seq Code for \( t \) Odd:

In this subsection we provide an upper bound on the rate of an \((n, k, r, t)\) seq code whenever \( t \) is a positive odd integer and \( r \geq 3 \).
Theorem 2. Let $C$ be an $(n,k,r,t)_{\text{seq}}$ code over a field $\mathbb{F}_q$. Let $t = 2s - 1$, $s \geq 1$ and $r \geq 3$. Then
\[
\frac{k}{n} \leq \frac{r^s}{r^s + 2 \sum_{i=1}^{s-1} r^i + 1}.
\] (26)

Proof. As earlier, we again begin by setting $B_0 = \text{span} \{ \xi \in C^\perp : w_H(\xi) \leq r + 1 \}$. Let $m$ be the dimension of $B_0$. Let $\xi_1, \ldots, \xi_m$ be a basis of $B_0$ such that $w_H(\xi_i) \leq r + 1$.

Let $H_1 = \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_m \end{bmatrix}$.

It follows that $H_1$ is a parity check matrix of an $(n,k,r,t)_{\text{seq}}$ code as its row space contains all the codewords of Hamming weight at most $r + 1$ which are contained in $C^\perp$. Also,
\[
\frac{k}{n} \leq 1 - \frac{m}{n}.
\]

We now proceed to derive an upper bound on $1 - \frac{m}{n}$. Without loss of generality the matrix $H_1$, after permutation of rows and columns can be written in the following form
\[
H_1 = \begin{bmatrix}
D_0 & A_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & D_1 & A_2 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & D_2 & A_3 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & D_3 & \cdots & 0 & 0 & 0 \\
& \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & A_{s-2} & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & D_{s-2} & A_{s-1} & 0 \\
0 & 0 & 0 & 0 & \cdots & D_{s-2} & A_{s-1} & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & C
\end{bmatrix},
\] (27)

where
1) the rows of $H_1$ are labeled $1, 2, \ldots, m$ and columns are labeled $1, 2, \ldots n$,
2) $A_i$ is a $\rho_{i-1} \times a_i$ matrix for $1 \leq i \leq s - 1$, $D_i$ is a $\rho_i \times a_i$ matrix for $0 \leq i \leq s - 1$ for some $\{\rho_i\}, \{a_i\}$.
3) $D_0$ is a matrix with each column having weight 1 and each row having weight at least 1.
4) $\{A_1, \{D_1, \{B_i = \begin{bmatrix} A_i \\ D_i \end{bmatrix} \} \}$ are such that for $1 \leq i \leq s - 1$ each column of $B_i$ has weight 2, each column of $A_i$ has weight at least 1, each row of $D_i$ has weight at least 1 and each column of $D_i$ has weight at most 1,
5) $C$ is a matrix with each column having weight 2, $D$ is a matrix which exactly contains all the columns of $H_1$ which have weight at least 3,
6) If $J$ is the first index such that $A_J, D_J (J = 0$ if $D_0$ is an empty matrix) are empty matrices (0 x $L$, $L$ x 0, 0 x 0 matrix) then we set $A_i, D_i$ to be empty matrices for all $s - 1 \geq i \geq J$ and set $a_j = 0, \rho_i = 0, s - 1 \geq i \geq J$, and place all of the 2-weight columns apart from the 2-weight columns corresponding to $B_1$ to $B_{J - 1}$ in $C$. Let the number of columns in $C$ be $a_s$. If $C$ has no columns then we set $a_s = 0$.

The entire rate bound derivation is correct and all the bounds will hold with $a_i = 0, \rho_i = 0, J \leq i \leq s - 1$. If $l (< J)$ is the first index such that $D_l$ is an empty matrix with $A_l$ a non empty matrix then the proof of the following claim \cite{2} (in the proof of claim \cite{2} we prove the claim for $A_l$ first and then proceed to $D_l$ and since $D_0, A_J, D_J$ must be non empty $\forall J < l$) will imply that $A_l$ has column weight 1 which will imply $D_l$ cannot be empty. Hence the case $A_l$, a non empty matrix and $D_l$, an empty matrix cannot occur. Although we have to prove the following claim \cite{2} for $A_i, D_i, 1 \leq i \leq J - 1, D_0$, we assume all $D_0, A_1, D_1, 1 \leq i \leq s - 1$ to be non-empty matrices and prove the claim. Since the proof is by induction, the induction can be made to stop at $J - 1$ (induction starts at 0) and the proof is unaffected by it.

Claim 2. For $1 \leq i \leq s - 1$, $A_i$ is a matrix with each column having weight 1 and for $0 \leq i \leq s - 2$, $D_i$ is a matrix with each row and each column having weight 1. $D_{s-1}$ is a matrix having column weight 1.

Proof. Proof is exactly similar to the proof of claim \cite{2} So we skip the proof. \hfill \Box

By Claim \cite{2} after permutation of columns of $H_1$ in (27) within the columns labeled by the set $\{\sum_{i=0}^{j-1} a_i + 1, \ldots, \sum_{i=0}^{j-1} a_i + a_j\}$ for $0 \leq j \leq \text{min}(J - 1, s - 2)$, the matrix $D_j, 0 \leq j \leq \text{min}(J - 1, s - 2)$ can be assumed to be a diagonal matrix with non-zero entries along the diagonal and hence $\rho_i = a_i$ for $0 \leq i \leq s - 2$. By counting the row weights and column weights of $A_i, 1 \leq i \leq s - 1$ we get (Note that if $A_j$ is an empty matrix then also the following inequality is true as we would have
set $a_j = 0$):

$$\rho_i - r \geq a_i,$$
$$a_i - r \geq a_i.$$  \hspace{1cm} (28)

For some $p \geq 0$,

$$\rho_{i-1} + \sum_{i=0}^{s-2} a_i + p = m.$$  \hspace{1cm} (29)

Counting the row weights and column weights of the matrix $[D_{s-1} | C]$, we get (Note that if $D_{s-1}$ or $C$ is an empty matrix then also the following inequality is true as we would have set $a_{s-1} = 0$ or $a_s = 0$ respectively):

$$2a_s + a_{s-1} \leq (\rho_{s-1} + p)(r + 1).$$  \hspace{1cm} (30)

Substituting (29) in (30):

$$2a_s \leq (m - \sum_{i=0}^{s-2} a_i)(r + 1) - a_{s-1}.$$  \hspace{1cm} (31)

Counting the row weights and column weights of $D_{s-1}$ (Note that if $D_{s-1}$ is an empty matrix then also the following inequality is true as we would have set $a_{s-1} = 0$):

$$a_{s-1} \leq \rho_{s-1}(r + 1).$$  \hspace{1cm} (32)

Counting the row weights and column weights of $H_1$, we get:

$$m(r + 1) \geq a_0 + 2(\sum_{i=1}^{s} a_i) + 3(n - \sum_{i=0}^{s} a_i),$$
$$m(r + 1) \geq 3n - 2a_0 - (\sum_{i=1}^{s} a_i).$$  \hspace{1cm} (33)

Our basic inequalities are (28), (29), (30), (32), (33). We manipulate these 5 inequalities to derive the bound on rate.

Substituting (31) in (33):

$$m(r + 1) \geq 3n - 2a_0 - (\sum_{i=1}^{s-1} a_i) = \left(\frac{(m - \sum_{i=0}^{s-2} a_i)(r + 1) - a_{s-1}}{2}\right).$$  \hspace{1cm} (34)

For $s = 1$, (35) becomes:

$$m(r + 1) \geq 3n - 2a_0 - \left(\frac{m(r + 1) - a_0}{2}\right),$$
$$m \frac{3(r + 1)}{2} \geq 3n - \frac{3}{2}a_0.$$  \hspace{1cm} (36)

Substituting (33) in (36):

$$m \frac{3(r + 1)}{2} \geq 3n - \frac{3}{2}m(r + 1).$$  \hspace{1cm} (37)

Substituting (29) in (37):

$$m \frac{3(r + 1)}{2} \geq 3n - \frac{3}{2}(m - p)(r + 1),$$
Since $p \geq 0$, $3m(r + 1) \geq 3n.$  \hspace{1cm} (38)

(38) implies,

$$\frac{k}{n} \leq \frac{r}{r + 1}.$$  \hspace{1cm} (39)

(39) proves the bound (26) for $s = 1$. Hence from now on we assume $s \geq 2$. 
For $s \geq 2$, (35) implies:

$$m \frac{3(r+1)}{2} \geq 3n + a_0 \left( \frac{r+1}{2} - 2 \right) + \sum_{i=1}^{s-2} a_i \left( \frac{r+1}{2} - 1 \right) - \frac{a_{s-1}}{2}. \tag{40}$$

Substituting (28) in (40) and since $r \geq 3$:

$$m \frac{3(r+1)}{2} \geq 3n + \frac{a_{s-1}}{r^{s-1}} \left( \frac{r+1}{2} - 2 \right) + \sum_{i=1}^{s-2} \frac{a_{s-1}}{r^{s-1-i}} \left( \frac{r+1}{2} - 1 \right) - \frac{a_{s-1}}{2},$$

$$m \frac{3(r+1)}{2} \geq 3n + a_{s-1} \left( \frac{1}{r} \right) \left( \frac{r+1}{2} - 1 \right) - \frac{1}{r^{s-1-1}} - \frac{1}{2},$$

$$m \frac{3(r+1)}{2} \geq 3n - a_{s-1} \left( \frac{3}{2r^{s-1}} \right). \tag{41}$$

Rewriting (29):

$$\rho_{s-1} + \sum_{i=0}^{s-2} a_i + p = m. \tag{42}$$

Substituting (33), (28) in (42):

$$\rho_{s-1} + \sum_{i=0}^{s-2} a_i + p = m,$$

$$\frac{a_{s-1}}{r+1} + \sum_{i=0}^{s-2} \frac{a_{s-1}}{r^{s-1-i}} \leq m - p,$$

$$a_{s-1} \leq \frac{m - p}{\frac{1}{r+1} + \sum_{i=1}^{s} \frac{1}{r^i}},$$

$$a_{s-1} \leq \frac{(m - p)(r+1)}{1 + \frac{(r^{s-1-1})(r+1)}{(r^{s-1})(r-1)}}. \tag{43}$$

Substituting (43) in (41):

$$m \frac{3(r+1)}{2} \geq 3n - \frac{(m - p)(r+1)}{1 + \frac{(r^{s-1-1})(r+1)}{(r^{s-1})(r-1)}} \left( \frac{3}{2r^{s-1}} \right),$$

$$\text{Since } p \geq 0, m \frac{3(r+1)}{2} \left( 1 + \frac{1}{r^{s-1} + rac{(r^{s-1-1})(r+1)}{(r^{s-1})(r-1)}} \right) \geq 3n. \tag{44}$$

After some algebraic manipulations gives the required bound on $1 - \frac{m}{n}$ and hence on $\frac{x}{n}$ as stated in the theorem. \qed

Again an alternative proof for Theorem 2 is given below by using linear programming:

Proof. The inequalities (28), (30), (33) and (34) are linear inequalities and are written in matrix form as:

$$Ax \geq b$$

where

$$A = \begin{bmatrix} r & -1 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 \\ 0 & r & -1 & \ldots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & r & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \ldots & 0 & -1 & (r+1) & -2 & (r+1) \\ 0 & 0 & 0 & \ldots & 0 & -1 & (r+1) & 0 & 0 \\ (r+3) & (r+2) & (r+2) & \ldots & (r+2) & 1 & (r+1) & 0 & (r+1) \end{bmatrix}, \tag{45}$$

which is a $(s+2) \times (s+3)$ matrix and

$$x = \begin{bmatrix} a_0 & a_1 & \ldots & a_{s-1} & \rho_{s-1} & a_s & p \end{bmatrix}^T, \quad b = \begin{bmatrix} 0 & 0 & \ldots & 0 & 3n \end{bmatrix}^T \tag{46}$$
where $\mathbf{x}$ is a $(s+3)\times 1$ matrix and $\mathbf{b}$ is a $(s+2)\times 1$ matrix. The problem of finding an upper bound on rate of the code now becomes one of minimizing $m = \mathbf{c}^T \mathbf{x}$, which is a linear objective function where $\mathbf{c} = [1 \, 1 \, \ldots \, 1 \, 0 \, 1 \, 0 \, 1]^T$ is a $(s+3)\times 1$ matrix. Also by definition of $\mathbf{x}$, $\mathbf{x} \geq 0$. This is now in a standard form of a linear program formulation as:

\[
\begin{align*}
\text{minimize} \quad & \mathbf{c}^T \mathbf{x} \\
\text{s.t.} \quad & \mathbf{A} \mathbf{x} \geq \mathbf{b} \\
& \mathbf{x} \geq 0
\end{align*}
\]

The dual problem of the above is

\[
\begin{align*}
\text{maximize} \quad & \mathbf{b}^T \lambda \\
\text{s.t.} \quad & \mathbf{A}^T \lambda \leq \mathbf{c} \\
& \lambda \geq 0
\end{align*}
\]

We will solve the dual problem by writing it in standard minimize $-\mathbf{b}^T \lambda$ form and using the simplex method. Let us introduce slack variables $h_1, \ldots, h_{s+2}$ and re-write the constraints as

\[
\mathbf{B}v = \mathbf{c}, \quad v \geq 0,
\]

where

\[
\mathbf{B} = \begin{bmatrix}
 r & 0 & 0 & \ldots & 0 & 0 & 0 & (r+3) & 0 & 1 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\
-1 & r & 0 & \ldots & 0 & 0 & 0 & (r+2) & 0 & 0 & 1 & 0 & \ldots & 0 & 0 & 0 & 0 \\
0 & -1 & r & \ldots & 0 & 0 & 0 & (r+2) & 0 & 0 & 0 & 1 & \ldots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & r & 0 & 0 & (r+2) & 0 & 0 & 0 & 0 & \ldots & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & \ldots & -1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & \ldots & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & \ldots & 0 & (r+1) & (r+1) & (r+1) & 0 & 0 & 0 & 0 & \ldots & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & \ldots & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & \ldots & 0 & (r+1) & 0 & (r+1) & 1 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0
\end{bmatrix},
\]

(47)

and

\[
\mathbf{v} = \begin{bmatrix}
 \lambda_1 & \lambda_2 & \ldots & \lambda_{s+2} & h_1 & \ldots & h_{s+3}
\end{bmatrix}^T.
\]

(48)

With this, the objective function now is $\mathbf{d}^T \mathbf{v}$, where $\mathbf{d} = [\mathbf{c}^T \, 0 \, 0 \, \ldots \, 0]^T$. We pick the variables $\lambda_1, \ldots, \lambda_{s+2}, h_1$ as “basic variables” and the rest, called “non-basic variables” will be set to 0. A set of basic variables is chosen such that the columns of $\mathbf{B}$ corresponding to those variables is a full-rank square matrix. The system of equations is now in the following form:

\[
\begin{bmatrix}
 \mathbf{B}v \\
\mathbf{B}_N \mathbf{v}
\end{bmatrix} \begin{bmatrix}
 \lambda_{BV} \\
0
\end{bmatrix} = \mathbf{c}
\]

Therefore we will equivalently solve

\[
\mathbf{B}_B \mathbf{v} = \mathbf{c}
\]

The above system of equations can be solved in closed form to get the following:

\[
\begin{align*}
\lambda_{s+2} &= \frac{2 \sum_{i=1}^{r-1} r^i + 1}{3(r^s + 2 \sum_{i=1}^{s-1} r^i + 1)}, \\
h_1 &= \frac{\sum_{i=1}^{s-1} r^i + 2}{3(\sum_{i=0}^{s-1} r^i)}, \\
\lambda_{j+1} &= \frac{r^s - 3r^{s-n-1} + r + 1}{3(r+1)(r^s - 1)}, \quad \text{for } 0 \leq j \leq s - 2 \\
\lambda_s &= 0, \\
\lambda_{s+1} &= \frac{\sum_{i=1}^{s-1} r^i + 2}{3(r+1)(\sum_{i=0}^{s-1} r^i)},
\end{align*}
\]

which are non-negative if $r \geq 3$. Hence the chosen basic solution is a basic feasible solution. To check for optimality we check if the “reduced cost coefficients” $r_{\alpha} = b_{\alpha} - z_{\alpha}$ are non-negative, for every non-basic variable $\alpha_i$. We note that for the above made choice of non-basic variables, $b_\beta = 0$. Suppose the non-basic variables are labeled $\alpha_1, \ldots, \alpha_M$ and the basic
variables are labeled $\beta_1, \ldots, \beta_N$, $z_{\alpha_i}$ is defined as follows:

$$z_{\alpha_i} = \sum_{\beta = \beta_i}^{\beta_N} b_{\beta} y(\beta, \alpha_i) = b_{s+2} y(s+2, \alpha_i) = -3ny(s+2, \alpha_i)$$

where $y(s+2, \alpha_i)$ are as shown in the row reduced echelon form of matrix $B$ below:

$$B_{ref} = \begin{bmatrix}
1 & 0 & 0 & \ldots & 0 & 0 & 0 & y(1, \alpha_1) & y(1, \alpha_2) & y(1, \alpha_3) & \cdots & y(1, \alpha_{M-1}) & y(1, \alpha_M) \\
0 & 1 & 0 & \ldots & 0 & 0 & 0 & y(2, \alpha_1) & y(2, \alpha_2) & y(2, \alpha_3) & \cdots & y(2, \alpha_{M-1}) & y(2, \alpha_M) \\
0 & 0 & 1 & \ldots & 0 & 0 & 0 & y(3, \alpha_1) & y(3, \alpha_2) & y(3, \alpha_3) & \cdots & y(3, \alpha_{M-1}) & y(3, \alpha_M) \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 1 & 0 & 0 & y(s+1, \alpha_1) & y(s+1, \alpha_2) & y(s+1, \alpha_3) & \cdots & y(s+1, \alpha_{M-1}) & y(s+1, \alpha_M) \\
0 & 0 & 0 & \ldots & 0 & 1 & 0 & y(s+2, \alpha_1) & y(s+2, \alpha_2) & y(s+2, \alpha_3) & \cdots & y(s+2, \alpha_{M-1}) & y(s+2, \alpha_M) \\
0 & 0 & 0 & \ldots & 0 & 0 & 1 & y(s+3, \alpha_1) & y(s+3, \alpha_2) & y(s+3, \alpha_3) & \cdots & y(s+3, \alpha_{M-1}) & y(s+3, \alpha_M)
\end{bmatrix}$$

We observe that in reducing row-$(s+2)$ we add only non-negative linear combinations of the rows above it, entries of which are either 0 or 1. Therefore $r_{\alpha_i} \geq 0$ for $\alpha_i$ all non-basic variables. Hence $y(s+2, \alpha_1), \ldots, y(s+2, \alpha_M) \geq 0$. Hence the chosen basic solution is an “optimal basic feasible” solution.

By the theorem of strong duality the optimal solutions of the primal problem and the dual problem are equal. Therefore the minimum value of $m$ is

$$m = \frac{n(2\sum_{i=1}^{s-1} r^i + 1)}{(r^s + 2\sum_{i=1}^{s-1} r^i + 1)}.$$ 

Hence we get the upper bound on the rate:

$$\frac{k}{n} \leq 1 - \frac{m}{n} = \frac{r^s}{(r^s + 2\sum_{i=1}^{s-1} r^i + 1)},$$

We now pick a solution for the primal problem and show that it is feasible and gives the optimal objective function value.

$$a_i = \frac{nr^i(r + 1)}{r^s + 2\sum_{i=1}^{s-1} r^i + 1}, \text{ for } 0 \leq i \leq s - 1$$

$$\rho_{s-1} = \frac{nr^{s-1}}{r^s + 2\sum_{i=1}^{s-1} r^i + 1}, \text{ } a_s = 0, \text{ } p = 0.$$

It is easy to check that this solution satisfies the constraints of the primal problem with equality. Therefore the chosen solution is a feasible solution. It is also easy to check that the solution gives the optimal value of the objective function. Hence it is an optimal feasible solution. We thus conclude that a code having the above chosen value could have the optimal rate.

From the two derivations, it becomes clear that a code achieving the rate bound given in (1) will have parity check matrix of the form (2) with $D$ being an empty matrix and all the inequalities (3), (4), (5), (7) met with equality. A similar observation holds for $t$ odd also. It may be noted here that our bound, for the special cases of $t = 2, 3, 4$, matches with the rate bound given in [8], [12], [11] respectively. In the rest of the paper, the codes achieving the bounds (1) or (26) depending on $t$ will be referred to as “rate-optimal codes”.

**Remark 1.** We now make a remark on the blocklength of the rate-optimal codes. From the proofs it becomes apparent that, for $t$ even, for the optimal values of $a_0, \ldots, a_{r}^2$ to be integral, $2n$ needs to be an integer multiple of $r_{2}^2 + 2\sum_{i=0}^{r-1} r^i$. A similar observation can be made for odd $t$.

**Remark 2.** In general, one would expect an $(n, k, r, t)_{\text{seq}}$ code to achieve a higher rate than a counterpart code having $t$-availability. The achievable upper bound on rate of a code with sequential recovery is significantly larger than the upper bound on the rate of codes having availability given in [10] as shown in Fig. 2.

### III. CONSTRUCTION OF CODES ACHIEVING THE UPPER BOUND ON RATE

In this section we give a construction of codes with sequential recovery achieving the rate bound (1) for any $r \geq 3, t = 2s, s \geq 1$ and also give a construction achieving (26) for any $r \geq 3, t = 2s - 1, s \geq 1$.

**A. Construction of codes achieving the upper bound on rate for $t$ even:**

In this subsection we give a construction of codes achieving the rate bound (1) for any $r \geq 3$ and $t \in 2\mathbb{Z}_+$. A construction achieving the bound (1) for the special case of $t = 2$ was provided in [8], [10], [11] and for the case of $t = 4$ was provided in [12].

We give a graph-based construction. We give the general principle of the construction as follows:
Consider a regular graph $G_0$ with $\hat{L}$ nodes each of degree $r$ and girth at least $t+1$. Let the nodes of $G_0$ be $\{p_1^0, ..., p_L^0\}$. Let $\hat{L}$ be a multiple of $r^{\frac{1}{r} - 1}$. If not we can take disjoint copies of $G_0$ and take disjoint union of these copies of $G_0$ such that the resulting graph has number of nodes which is a multiple of $r^{\frac{1}{r} - 1}$. Partition the nodes $\{p_1^0, ..., p_L^0\}$ into $\frac{L}{r}$ sets with each set containing exactly $r$ nodes. Let us denote the partition as $\{Q_1^0, ..., Q_{\frac{L}{r}}^0\}$.

Now add a set of $\frac{L}{r}$ nodes denoted by $\{p_1^1, ..., p_{\frac{L}{r}}^1\}$ to $G_0$. Connect the node $p_j^1$ to the nodes in $Q_j^0$ for $1 \leq j \leq \frac{L}{r}$. Let the resulting graph be $G_1$.

Now we inductively construct the graph $G_i$ from $G_{i-1}$ for $0 < i \leq \frac{L}{r} - 1$.

Nodes are added to $G_{i-1}$, for $0 < i \leq \frac{L}{r} - 1$ to form $G_i$ as follows: partition the nodes $\{p_1^{i-1}, ..., p_{\frac{L}{r}}^{i-1}\}$ into $\frac{L}{r}$ sets with each set containing exactly $r$ nodes. Let the partition be $\{Q_1^{i-1}, ..., Q_{\frac{L}{r}}^{i-1}\}$. Add new nodes labeled $\{p_1^i, ..., p_{\frac{L}{r}}^i\}$ to $G_{i-1}$. Now connect $p_j^i$ to $r$ nodes in the set $Q_j^{i-1}$ for $1 \leq j \leq \frac{L}{r}$. The resulting graph is denoted as $G_i$.

We now define our code based on $G_{\frac{L}{r}-1}$. Let the edges of $G_0$ (viewed as subgraph of $G_{\frac{L}{r}-1}$) represent information symbols and the nodes $\cup_{i=0}^{\frac{L}{r}-1}\{p_1^i, ..., p_{\frac{L}{r}}^i\}$ represent parity symbols. The node $p_j^0$ represents a code symbol storing the parity of the information symbols that are represented by edges incident on it in $G_0$. For $0 < i \leq \frac{L}{r} - 1$, the node $p_j^i$ represent a code symbol storing the parity of code symbols represented by the nodes in the set $Q_j^{i-1}$. The resulting code $C$ is defined by the information symbols represented by the edges in $G_0$ and the parity symbols represented by $\cup_{i=0}^{\frac{L}{r}-1}\{p_1^i, ..., p_{\frac{L}{r}}^i\}$.

If we construct $G_i$ from $G_{i-1}$ in such a way that the girth of $G_i$ is at least $t+1$, then it is clear that the resulting code $C$ is an $(n, k, r, t)_{seq}$ code with $k = \frac{L^2}{r}$ and $n = k + \sum_{i=0}^{\frac{L}{r}-1} \frac{L}{r}$. Hence the code $C$ has rate achieving the upper bound given in [1].

Hence it is enough to give a construction for $G_i$ from $G_{i-1}$, for $1 \leq i \leq \frac{L}{r} - 1$ such that girth of $G_{i-1}$ is at least $t+1$. We construct $G_0$ separately as step 0.

It can be noted that the graph $G_{\frac{L}{r}-1}$ after removing the edges in $G_0$ has a tree structure with nodes $\{p_j^i : 1 \leq j \leq \hat{L}\}$ at level $i$ (we represent the last level as level 0) with nodes $\{p_j^0 : 1 \leq j \leq \hat{L}\}$ as leaf nodes.

B. Construction of $G_i$:

At Step 0: $G_0$ is simply a regular graph of degree $r$ with girth at least $t+1$ which can be constructed [20].

Let us assume we are given the graph $G_{i-1}$ for some $1 \leq i \leq \frac{L}{r} - 1$ with girth at least $t+1$ with nodes labeled as before constructed at step $i-1$. Let us construct $G_i$ as follows.

Step i:

Take a biregular bipartite graph $G' = (V_1 \cup V_2, E)$ with $V_1 = \{v_1, ..., v_m\}, V_2 = \{w_1, ..., w_n\}$ and $\text{degree}(v_j) = r, 1 \leq j \leq m, \text{degree}(w_j) = \frac{\hat{L}}{r}, 1 \leq j \leq n_i$ with girth at least $\lceil \frac{t+1}{r} \rceil$. Such a graph can be constructed [20]. Now $m = n_i \frac{L}{r}$.

Construct a new graph $G''$ as follows:

- In $G'$, replace the node $w_f$ by $\{w_{1,f}, ..., w_{\frac{L}{r},f}\}$ for $1 \leq f \leq n_i$. If the neighbours of $w_f$ in $G'$ are the nodes in $\{v_{q_1}, ..., v_{q_{\frac{L}{r}}}\}$, then connect $w_{j,f}$ and $v_{q_j}$ by an edge for $1 \leq j \leq \frac{L}{r}$. The resulting graph is termed as $G''$. 

![Fig. 2: Comparison of rate bounds on codes with availability and codes with sequential recovery for an example case $t = 10$.](image)
Now take $n_t$ disjoint copies of the graph $G_{i-1}$, and form the graph $G'_{i-1}$ with the disjoint union of these $n_t$ copies of $G_{i-1}$. Note that $G'_{i-1}$ can be constructed in exactly the same way $G_{i-1}$ is constructed starting from $n_t$ disjoint copies of $G_0$ and girth of $G'_{i-1}$ can be seen to be at least $t + 1$. Hence we take $G'_{i-1}$ as our graph construction at $(i - 1)$th step.

Let the nodes in $G_{i-1}$ corresponding to the $n_t$ disjoint copies of $p_{j\rightarrow}^{i-1}, 1 \leq j \leq \frac{n_t}{r}$ in disjoint copies of $G_{i-1}$ be labeled as $p_{j\rightarrow}^{i-1}$ for $1 \leq j \leq \frac{n_t}{r}$, where $p_{j\rightarrow}^{i-1}$ corresponds to the nodes $p_{j\rightarrow}^{i-1}, 1 \leq j \leq \frac{n_t}{r}$ in the $f$th copy of $G_{i-1}$ respectively, for $1 \leq f \leq n_t$.

Now take disjoint union of $G'$ and $G'_{i-1}$. Merge the node $w_{j,f}$ with $p_{j\rightarrow}^{i-1}$ for $1 \leq f \leq n_t, 1 \leq j \leq \frac{n_t}{r}$. Relabel $v_j$ as $p_{j\rightarrow}^{i}$. The resulting graph is our desired graph $G_i$.

Since the girth of $G'_{i-1}$ is at least $t + 1$ and girth of $G'$ is at least $\lceil \frac{t+1}{i+\frac{1}{2}} \rceil$, it can be easily seen that the graph $G_i$ has girth at least $t + 1$.

Hence the bound given in (I) is tight and achievable for any $r \geq 3$ and $t \in 2\mathbb{Z}_+$. 

**C. Construction of codes achieving the upper bound on rate for $t$ odd:**

In this subsection we give a construction of codes achieving the rate bound (26) for any $r \geq 3$ and $t = 2s - 1, s \geq 1$. A construction achieving the bound (26) for $t = 5$ can be found in [12]. For $t = 3$, the bound (26) can be achieved by taking the product of two $[r + 1, r]$ single parity check code (2 dimensional product code).

We give a graph-based construction. We give the general principle of the construction as follows:

- Consider a regular bipartite graph $G_0 = (V_1 \cup V_2, E)$ with $V_1 = \{p_1^0, ..., p_L^0\}$ and $V_2 = \{u_1^0, ..., u_L^0\}$ and $\text{degree}(p_i^0) = \text{degree}(u_i^0) = r$ for $1 \leq i \leq L$. We pick $G_0$ such that its girth is at least $t + 1$. Let $L$ be a multiple of $rs^{-1}$. If not we can take disjoint copies of $G_0$ and take disjoint union of these copies of $G_0$ such that the resulting graph has number of nodes which is a multiple of $rs^{-1}$.

- Partition the nodes in $V_1$ into $\frac{L}{r}$ sets with each set containing exactly $r$ nodes. Let the partition be $\{Q_1^0, ..., Q_{\frac{L}{r}}^0\}$. Similarly partition the nodes in $V_2$ into $\frac{L}{r}$ sets with each set containing exactly $r$ nodes. Let the partition be $\{S_1^0, ..., S_{\frac{L}{r}}^0\}$.

- Add new nodes $\{p_1^1, ..., p_{\frac{L}{r}}^1\} \cup \{u_1^1, ..., u_{\frac{L}{r}}^1\}$ to $G_0$. Connect $p_j^1$ to the $r$ nodes in $Q_j^0$ for $1 \leq j \leq \frac{L}{r}$. Similarly connect $u_j^1$ to the $r$ nodes in $S_j^0$ for $1 \leq j \leq \frac{L}{r}$. Let the resulting graph be $G_1$.

- Now we will inductively construct a graph $G_i$ from $G_{i-1}$, for $1 \leq i \leq s - 1$.

- We will construct $G_i$ from $G_{i-1}$ as follows:

  - Partition the nodes $\{p_{1\rightarrow}^{i-1}, ..., p_{\frac{L}{r}\rightarrow}^{i-1}\}$ into $\frac{L}{r}$ sets with each set containing exactly $r$ nodes. Let the partition be $\{Q_1^{i-1}, ..., Q_{\frac{L}{r}}^{i-1}\}$. Similarly partition the nodes $\{u_{1\rightarrow}^{i-1}, ..., u_{\frac{L}{r}\rightarrow}^{i-1}\}$ into $\frac{L}{r}$ sets with each set containing exactly $r$ nodes. Let the partition be $\{S_1^{i-1}, ..., S_{\frac{L}{r}}^{i-1}\}$.

  - Add new nodes $\{p_1^i, ..., p_{\frac{L}{r}}^i\} \cup \{u_1^i, ..., u_{\frac{L}{r}}^i\}$ to $G_{i-1}$. Connect $p_j^i$ to the $r$ nodes in $Q_j^{i-1}$ for $1 \leq j \leq \frac{L}{r}$. Similarly connect $u_j^i$ to the $r$ nodes in $S_j^{i-1}$ for $1 \leq j \leq \frac{L}{r}$. The resulting graph is denoted as $G_i$.

- Follow the above procedure and construct $G_{s-1}$. We now remove the nodes $\{u_1^{s-1}, ..., u_{\frac{L}{r}}^{s-1}\}$ and the edges incident on them from $G_{s-1}$. We call the resulting graph $G'_{s-1}$. We now define our code based on $G'_{s-1}$.

- Let the edges of $G_0$ (viewed as subgraph of $G'_{s-1}$) represent information symbols and the nodes $\cup_{i=0}^{s-1}\{p_1^i, ..., p_{\frac{L}{r}}^i\} \cup \cup_{i=0}^{t-2}\{u_1^i, ..., u_{\frac{L}{r}}^i\}$ represent parity symbols. The nodes $p_j^0$ and $u_j^0$ represent code symbols storing the parity of the information symbols that are represented by edges incident on them in $G_0$. For $0 < i < s - 1$, the node $p_j^i$ represents a code symbol storing the parity of code symbols represented by the nodes in the set $Q_j^{i-1}$ and similarly for $0 < i < s - 1$, $u_j^i$ represents a code symbol storing the parity of code symbols represented by the nodes in the set $S_j^{i-1}$. The resulting code $C$ is defined by the information symbols represented by the edges in $G_0$ and the parity symbols represented by $\cup_{i=0}^{s-2}\{p_1^i, ..., p_{\frac{L}{r}}^i\} \cup \cup_{i=0}^{t-2}\{u_1^i, ..., u_{\frac{L}{r}}^i\}$.

- If we construct $G_i$ from $G_{i-1}$ in such a way that the girth of $G_i$ is at least $t + 1$, then it is clear that the resulting code $C$ is an $(n, k, r, t)_{\text{seq}}$ code with $k = Lr$ and $n = k + 2\sum_{i=0}^{s-2} \frac{r}{r - i} + \frac{L}{r - s}$. Hence the code $C$ has rate achieving the upper bound given in (26).

- Hence it is enough to give a construction for $G_i$ from $G_{i-1}$, for $1 \leq i \leq \frac{1}{2} - 1$ such that girth of $G_i$ is at least $t + 1$ assuming girth of $G_{i-1}$ is at least $t + 1$. We construct $G_0$ separately as step 0.

- It can be noted that the graph $G_{s-1}$ after removing the edges in $G_0$ has a tree structure with nodes $\{p_j^i : 1 \leq j \leq \frac{L}{r}\} \cup \{u_j^i : 1 \leq j \leq \frac{L}{r}\}$ at level $i$ (we represent the last level as level 0) with nodes $\{p_j^0 : 1 \leq j \leq L\} \cup \{u_j^0 : 1 \leq j \leq L\}$ as leaf nodes.
D. Construction of $G_i$:

- At Step 0: $G_0$ is simply a regular bipartite graph of degree $r$ with girth at least $t+1$ which can be constructed \cite{20}.
- Let us assume we are given the graph $G_{i-1}$ for some $1 \leq i \leq s-1$ with girth at least $t+1$ with nodes labeled as before constructed at step $i-1$. Let us construct $G_i$ as follows. 

Step i:

- Take a biregular bipartite graph $G' = (V_3 \cup V_4, E)$ with $V_3 = \{v_1, ..., v_{m_3}\}$ and $V_4 = \{w_1, ..., w_{m_4}\}$ and degree($v_j$) = $r$ for $1 \leq j \leq 2m$ and degree($w_j$) = \(\frac{2r}{r+1}\) for $1 \leq j \leq n_i$. Let $N(v)$ represent the neighbours of the node $v$. We pick $G'$ such that $|N(w_j) \cap \{v_1, ..., v_{m_3}\}| = |N(w_j) \cap \{v_{m_1+1}, ..., v_{2m}\}| = \frac{r}{r+1}$, for $1 \leq j \leq n_i$. We also pick $G'$ such that its girth is at least $\left\lceil \frac{t+1}{r+1} \right\rceil$. We will now assume that $G'$ with the mentioned properties can be constructed and construct $G_i$ using $G'$ and $G_{i-1}$. We will separately describe the construction of $G'$ with the mentioned properties.

- Construct a new graph $G''$ as follows:
  
  - In $G''$, replace the node $w_j$, by $\{w_{1,f}, ..., w_{\frac{2r}{r+1},f}\}$ for $1 \leq f \leq n_i$. If the neighbours of $w_j$ in $G'$ are nodes in the set $\{v_{q_1}, ..., v_{\frac{2r}{r+1},q}\} \cup \{v_{q_2,\frac{2r}{r+1}+1}, ..., v_{q_2,2}\}$ with $\{v_{q_1,1}, ..., v_{q_2,\frac{2r}{r+1}}\} \subseteq \{v_1, ..., v_{m_3}\}$ and $\{v_{q_2,\frac{2r}{r+1}+1}, ..., v_{q_2,2}\} \subseteq \{v_{m_1+1}, ..., v_{2m}\}$, then connect $w_{j,f}$ and $v_{q_2}$ by an edge for $1 \leq j \leq \frac{2r}{r+1}$.

- Now take $n_i$ disjoint copies of the graph $G_{i-1}$, and form the graph $G'_{i-1}$ with the disjoint union of these $n_i$ copies of $G_{i-1}$. As explained in the general principle of construction, nodes in $G_{i-1}$ are represented by $\{\cup_{j=0}^{n_i-1}\{p_{1,j}, ..., p_{j,\frac{2r}{r+1}}\} \cup \cup_{j=0}^{n_i-1}\{u_{1,j}, ..., u_{j,\frac{2r}{r+1}}\}\}$. Let the nodes in $G'_{i-1}$ corresponding to the $n_i$ disjoint copies of $\{p_{1,j}, ..., p_{j,\frac{2r}{r+1}}\}$ $\cup \{u_{1,j}, ..., u_{j,\frac{2r}{r+1}}\} \cup \cup_{j=0}^{n_i-1}\{l_{1,j}, ..., l_{j,\frac{2r}{r+1}}\}$. The resulting graph is termed as $G''$.

- Now take disjoint union of $G''$ and $G_{i-1}$. Merge the node $w_{j,f}$ with $\frac{2r}{r+1}+1 \leq j \leq n_i, 1 \leq f \leq \frac{2r}{r+1}$ in the $j$th copy of $G_{i-1}$ respectively, for $1 \leq f \leq n_i$. Similarly $w_{j,f}$, $\frac{2r}{r+1}+1 \leq j \leq n_i, 1 \leq f \leq \frac{2r}{r+1}$ correspond to the nodes $u_{j,1}, 1 \leq b \leq \frac{2r}{r+1}$ in the $j$th copy of $G_{i-1}$ respectively, for $1 \leq f \leq n_i$. Note that $G'_{i-1}$ can be constructed in exactly the same way $G_{i-1}$ is constructed starting from $G'_0 = (V_3 \cup V_4, E)$ which is disjoint union of $n_i$ disjoint copies of $G_0$ with $V_3 = \{p_{0,1}, ..., p_{0,\frac{2r}{r+1}}\}$. The resulting graph is our desired graph $G_i$.

- Since the girth of $G'_{i-1}$ is at least $t+1$ and girth of $G''$ is at least $\left\lceil \frac{t+1}{r+1} \right\rceil$, it can be easily seen that the graph $G_i$ has girth at least $t+1$.

- Hence the construction of $G_i$ is done if the construction of $G''$ with the mentioned properties.

Constructions of $G'$:

- Consider a biregular bipartite graph $G^1 = (U_1 \cup U_2, E)$ with $U_1 = \{b_1, ..., b_{m}\}$, $U_2 = \{c_1, ..., c_{\lambda}\}$ with degree($b_j$) = $r$, for $1 \leq j \leq m$ and degree($c_j$) = $\frac{2r}{r+1}$, for $1 \leq j \leq \lambda$. Pick the graph $G^1$ such that it has girth at least $\left\lceil \frac{t+1}{r+1} \right\rceil$.
  
  Such a graph $G^1$ can be constructed due to \cite{20}.

- Consider a regular bipartite graph $G^2 = (X \cup Y, E)$ with $X = \{x_1, ..., x_L\}$ and $Y = \{y_1, ..., y_L\}$ with each node of degree $\lambda$ and girth at least $\frac{1}{2}\left(\frac{t+1}{r+1}\right)$. Such a graph $G^2$ can be constructed due to \cite{20}.

- Take $2^L$ disjoint copies of $G^1$. Let $G^3$ be the disjoint union of the $2^L$ copies named $G^1_1, ..., G^1_{2^L}$ of $G^1$. Denote the nodes $b_j, c_j$ in the $j$th copy $G^1_i$ by $b^i_j, c^i_j$ respectively.

- Replace the node $x_f$ in $G^2$ by $x_{1,f}, ..., x_{L,f}$ for $1 \leq f \leq L$. If the neighbours of $x_f$ in $G^2$ are the nodes in the set $\{y_{1,f}, ..., y_{L,f}\}$ then connect $x_{1,f}$ to $y_{1,f}$ for $1 \leq f \leq L$. Now replace the node $y_f$ by $y_{1,f}, ..., y_{L,f}$ for $1 \leq f \leq L$. If the neighbours of $y_f$ after replacing $x_f$ were the nodes in the set $\{x_{1,1}, ..., x_{1,\lambda}\}$ then connect $x_{1,1}$ to $x_{1,1}$ for $1 \leq j \leq \lambda$. Denote this graph by $G^4$. Now, if $x_{1,1}, y_{1,1}$ are neighbours in $G^4$ then merge the nodes $c^{1,1}_j$ and $c^{1,1}_j$ in $G^4$ and label the resulting merged node as $w_{(f-1)\lambda+j}$ in $G^3$. Relabel $b^{1,1}_j$ as $v_{(t-1)\lambda+j}$ in $G^3$. The resulting graph is named $G'$.

- $G'$ is a biregular bipartite graph $G' = (V_3 \cup V_4, E)$ with $V_3 = \{v_1, ..., v_{2m}\}$ where $m = \lambda \cdot \Lambda$ and degree($v_j$) = $r$ for $1 \leq j \leq 2m$ and degree($w_j$) = $2r$ for $1 \leq j \leq n_i$. It can be seen that $G'$ has girth at least $\left\lceil \frac{t+1}{r+1} \right\rceil$. It can also be seen that $|N(w_j) \cap \{v_1, ..., v_{m}\}| = |N(w_j) \cap \{v_{m+1}, ..., v_{2m}\}| = \frac{r}{r+1}$, for $1 \leq j \leq n_i$.

E. An example construction for $t = 4$ and $r = 3$

- At step 0 we consider a regular graph of degree $r = 3$ and girth at least $t + 1 = 5$. Petersen graph is such a graph and is known to have girth 5. It has 10 vertices and 15 edges. As our construction requires the number of vertices $L$ to be a
multiple of $r^{\frac{1}{2} - 1} = 3$ we take a disjoint union of 3 copies of the Petersen graph as shown in figure 3. It is the graph $G_0$
with $L = 30$.

- In step 1 we pick a biregular bipartite graph $G' = (V_1 \cup V_2, E)$ with $V_1 = \{v_1, ..., v_m\}, V_2 = \{w_1, ..., w_{n_1}\}$ and
degree($v_j$) = 3, 1 $\leq$ $j$ $\leq$ $m$, degree($w_j$) = $\frac{L}{n_1}$ = 30, 1 $\leq$ $j$ $\leq$ $n_1$ with girth at least $\lceil \frac{t+1}{r+1} \rceil$. The smallest
value of $n_1$ for degree($v_j$) to be 3 is $n_1 = 3$. Then, $m = \frac{n_1 L}{r} = 30$. The graph $G'$ that we pick is shown in figure 4.

- We then construct graph $G''$ from $G'$ as shown in figure 5.

- We then take disjoint union of $n_1 = 3$ copies of $G_0$ and call the resulting graph $G'_0$. After the merger and relabeling
of nodes as said in the construction we get the complete construction as shown in figure 6. If two nodes are connected by
a dashed line the two nodes represent a single node formed by the merger.

The graph $G_1$ thus constructed describes the code construction for $t = 4$, $r = 3$. As described in the construction the
nodes of the graph corresponding to parity symbols. The total number of nodes in $G_1$ is $n - k = 120$. The edges in the
9 copies of the Petersen graph correspond to the information symbols. The total number of information symbols is
$k = 135$. Therefore the blocklength of the code is $n = 255$. Thus the rate of the code is $\frac{9}{17}$ which achieves the upper
bound given by 3 for $t = 4$ and $r = 3$.

We now give a parity check matrix for the example code construction:
The edges of the 9 copies of Petersen graph can be arbitrarily labeled. With one particular labeling we define the following
$(10 \times 15)$ matrix:

$$C_1 = \begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
\end{bmatrix}, \quad (49)$$

The complete parity check matrix, which is a $(120 \times 255)$ matrix is as follows:

$$H = \begin{bmatrix}
I_{30} & I_{30} & I_{30} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
I_{30} & I_{30} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}, \quad (50)$$

![Graph G0 for an example construction for t = 4, r = 3](image)

**Remark 3.** The following conjecture on the rate of an $(n, k, r, t)$ seq code appeared in [13]. Let $m = \lceil \log_r(k) \rceil$.

$$\frac{k}{n} \leq \frac{1}{1 + \sum_{i=1}^{m} \frac{a_i}{r^i}}, \text{ where } a_i \geq 0, a_i \in \mathbb{Z}, \sum_{i=1}^{m} a_i = t.$$
The rate bound derived in the present paper can be verified to prove the conjecture. (More specific conjectures were provided for \( t = 5, 6 \). While our bound proves the conjecture for \( t = 5 \), the conjectured upper bound for \( t = 6 \) does not hold as our upper bound on code rate is both larger and achievable).

**Remark 4.** The following conjecture on the rate of an \((n, k, r, t)\)seq code appeared in [13].

\[
\frac{k}{n} \leq \frac{1}{1 + \sum_{i=1}^{m} \frac{a_i}{r^i}}
\]

\( a_i \geq 0, a_i \in \mathbb{Z}, \sum_{i=1}^{m} a_i = t, \)

\( m = \lceil \log_r(k) \rceil. \)

Our rate bound verifies the conjecture as our rate bound can be written in the form:

For \( t \) even:

\[
\frac{k}{n} \leq \frac{1}{1 + \sum_{i=1}^{\frac{t}{2}} \frac{2}{r^i}}
\]

For \( t \) Odd:

\[
\frac{k}{n} \leq \frac{1}{1 + \sum_{i=1}^{s-1} \frac{2}{r^i} + \frac{1}{r^s}}
\]

More specific conjecture was given for \( t = 5, 6 \). While our bound proves the conjecture for \( t = 5 \), the conjectured upper bound for \( t = 6 \) does not hold as our upper bound on code rate is both larger and achievable).

For \( t = 6 \), Our bound takes the form:

\[
\frac{k}{n} \leq \frac{r^3}{r^3 + 2r^2 + 2r + 2}
\]

\[
\frac{k}{n} \leq \frac{1}{1 + \frac{2}{r} + \frac{2}{r^2} + \frac{2}{r^3}}
\]

The conjecture for \( t = 6 \) given in [13]:

\[
\frac{k}{n} \leq \frac{1}{1 + \frac{2}{r} + \frac{1}{r^2} + \frac{1}{r^3}}
\]
Fig. 6: Graph $G_1$ for an example construction for $t = 4$, $r = 3$ describing the full code

Hence the conjecture given in [13] has a smaller rate than our bound for $t = 6$. And further our upper bound on rate for $t = 6$ can be achieved.

Remark 5. In [9] the authors provide a construction of a code with sequential recovery for any $r$ and $t$ having rate $\frac{r - 1}{r + 1} + \frac{1}{n}$. Moore graphs (degree $r + 1$, girth $t + 1$) will meet our rate bound exactly when a Moore graph of degree $r + 1$ and girth $t + 1$ exists. Moore graphs with degree $r + 1$, girth $t + 1$ are shown to not exist for any $t \not\in \{2, 3, 4, 5, 7, 11\}$ for any $r \geq 2$ (see [21]). Hence the construction in [9] is not rate-optimal for most of the cases.

APPENDIX A

PROOF OF CLAIM 1

It’s enough to show that:
- $\{A_i\}$ are matrices with each column having Hamming weight 1.
- $\{D_i\}$ are matrices with each row having Hamming weight 1.

As the point 1 written above combined with the fact that each column of $[A_i D_i]$ has Hamming weight 2 implies $D_i, i \geq 1$ are matrices with each column having Hamming weight 1 and $D_0$ by definition is a matrix with each column having Hamming weight 1.

Let us show the claim by induction as follows:

Induction Hypothesis:
- Property $P_i$: any $m \times 1$ vector having Hamming weight at most 2 with support contained in $\{\sum_{l=0}^{j-1} \rho_l + 1, ..., \sum_{l=0}^{j-1} \rho_l + \rho_j\}$ for some $0 \leq j \leq i$ can be written as some linear combination of at most $2(i + 1)$ column vectors of $H_1$ with labels in $\{1, ..., \sum_{l=0}^{i} a_l\}$.
- The property $P_i$ is true and the Claim 1 is true for $A_1, ..., A_i, D_0, ..., D_i$.

Initial step:
- We show that each row of $D_0$ has Hamming weight exactly 1.
  Suppose there exists a row of $D_0$ with Hamming weight more than 1; let the support set of the row be $i_1, i_2, ...$. Then the columns labeled $i_1, i_2$ of $H_1$ can be linearly combined to give a zero column. This contradicts the fact that $d_{min}(C) \geq t + 1, t > 0$, even. Hence, all rows of $D_0$ have Hamming weight exactly 1.
- If $t = 2$, then the claim is already proved. So let $t \geq 4$.
- We show that columns of $A_1$ have Hamming weight exactly 1.
  Suppose $j^{th}$ column of $A_1$ for some $1 \leq j \leq a_1$ has Hamming weight 2; let the support of the column be $j_1, j_2$ in $A_1$. Then the column labeled $a_0 + j$ in $H_1$ along with the 2 columns vectors of $H_1$ with labels from the set $\{1, ..., a_0\}$ where one of the column vector has exactly one non-zero entry in $j_1$ and other in $j_2$ can be linearly combined to give a zero column again leading to a contradiction on minimum distance.
The above argument also shows that any $m \times 1$ vector having Hamming weight at most 2 with support contained in \{1, ..., $\rho_0$\} can be written as some linear combination of at most 2 column vectors of $H_1$ with labels from the set \{1, ..., $\alpha_0$\}. Hence Property $P_0$ is true.

We now show that each row of $D_1$ has Hamming weight exactly 1. Suppose $j^{th}$ row of $D_1$ has Hamming weight more than 1; let the support set of the row be \{l_1, l_2, ..., l_2\} in $D_1$. Now there is some linear combination of columns labeled $a_0 + l_1$ and $a_0 + l_2$ in $H_1$ that gives a zero in \($\rho_0 + j$\)th coordinate and thus the linear combination has support contained in \{1, ..., $\rho_0$\} with Hamming weight at most 2. Thus by Property $P_0$, there is a non-zero set of at most 4 linearly dependent columns in $H_1$ leading to a contradiction on minimum distance.

Now we show that Property $P_1$ is true: It is enough to prove that any $m \times 1$ vector with weight at most 2 with support contained in \{\$\rho_0 + 1, ..., \rho_0 + \rho_1$\} can be written as linear combination of at most 2\((1+1) = 4\) vectors of $H_1$ with labels in \{1, ..., $\sum_{i=0}^{\rho_1} \alpha_i$\}. This can be easily seen using arguments similar to ones presented before. Let an $m \times 1$ vector have non-zero entries in coordinates $\rho_0 + j_1, \rho_0 + j_2$ or $\rho_0 + j_1$. Then this vector can be linearly combined with 2 atmost column vectors in $H_1$ labeled in \{\$a_0 + 1, ..., a_0 + \alpha_1$\} to form a $m \times 1$ vector with Hamming weight atmost 2 with support contained in \{1, $\rho_0$\} which intern can be written as linear combination of atmost 2 column vectors in $H_1$ labeled in \{1, ..., $\alpha_0$\} by property $P_0$. Hence the given $m \times 1$ vector is written as linear combination of atmost 2\((1+1) = 4\) column vectors in $H_1$ labeled in \{1, ..., $\sum_{i=0}^{\rho_1} \alpha_i$\}.

Induction step:

Let us assume by induction hypothesis that Property $P_i$ is true and the Claim \[\] is true for $A_1, ..., A_i, D_0, ..., D_i$ for some $i \leq \frac{t}{2} - 2$ and prove the induction hypothesis for $i + 1$. For $t = 4$, the initial step of induction completes the proof of Claim \[\]. Hence assume $t > 4$.

Now we show that each column of $A_{i+1}$ has Hamming weight exactly 1: suppose $j^{th}$ column of $A_{i+1}$ for some $1 \leq j \leq a_{i+1}$ has Hamming weight 2; let the support of the column be $j_1, j_2$ in $A_{i+1}$. It is clear that the corresponding column vector in $H_1$ is a vector with support contained in \{\$\sum_{i=0}^{j_1} \rho_i + 1, ..., \sum_{i=0}^{j_2} \rho_i + \rho_1$\}. Hence by Property $P_i$, there is a non-zero set of at most $2(i + 1) + 1$ columns in $H_1$ which are linearly dependent; hence contradicts the minimum distance as $2(i + 1) + 1 \leq t - 1$. Hence each column of $A_{i+1}$ has Hamming weight exactly 1.

Now we show that each row of $D_{i+1}$ has Hamming weight exactly 1: suppose $j^{th}$ row of $D_{i+1}$ has Hamming weight more than 1; let the support set of the row be $l_1, ..., l_2$ in $D_{i+1}$. Now some linear combination of columns labeled $\sum_{j=0}^{i} a_j + l_1$ and $\sum_{j=0}^{i} a_j + l_2$ in $H_1$ will make the resulting vector have a 0 in \((\sum_{i=0}^{j} \rho_i) + \rho_1$\)th coordinate and the resulting vector also has Hamming weight atmost 2 with support contained in \{\$\sum_{i=0}^{j-1} \rho_i + 1, ..., \sum_{i=0}^{j-1} \rho_i + \rho_1$\} and hence by Property $P_i$, there is a non-zero set of at most $2(i + 1) + 2$ columns in $H_1$ which are linearly dependent; hence contradicts the minimum distance as $2(i + 1) + 2 \leq t$; thus proving that each row of $D_{i+1}$ has Hamming weight exactly 1.

Now we show that Property $P_{i+1}$ is true: It is enough to prove that any $m \times 1$ vector with weight at most 2 with support contained in \{\$\sum_{i=0}^{j} \rho_i + 1, ..., \sum_{i=0}^{j} \rho_i + \rho_{i+1}$\} can be written as linear combination of at most $2(i + 2)$ vectors of $H_1$ with labels in \{1, ..., $\sum_{i=0}^{j} \rho_i + \rho_{i+1}$\}. This can be easily seen using arguments similar to ones presented before. Let an $m \times 1$ vector have non-zero entries in coordinates $\sum_{i=0}^{j} \rho_i + j_1, \sum_{i=0}^{j} \rho_i + j_2$ or $\sum_{i=0}^{j} \rho_i + j_1$. Then this vector can be linearly combined with 2 atmost column vectors in $H_1$ labeled in \{\$\sum_{i=0}^{j} \rho_i + 1, ..., \sum_{i=0}^{j} \rho_i + \rho_{i+1}$\} to form a $m \times 1$ vector with Hamming weight atmost 2 with support contained in \{\$\sum_{i=0}^{j} \rho_i + 1, \sum_{i=0}^{j} \rho_i + \rho_1$\} which intern can be written as linear combination of atmost $2(i + 1)$ column vectors in $H_1$ labeled in \{1, ..., $\sum_{i=0}^{j} \alpha_i$\} by property $P_i$. Hence the given $m \times 1$ vector is written as linear combination of atmost $2(i + 2)$ column vectors in $H_1$ labeled in \{1, ..., $\sum_{i=0}^{j} \alpha_i$\}.

REFERENCES


