Sparse Sensing for Statistical Inference
Model-driven and data-driven paradigms

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Power networks, grid analytics

Radio astronomy (e.g., SKA)

Health informatics

Internet, social media

Design sparse sensing functions
What is sparse sensing?

Design \( w \in \{0, 1\}^D \) to select the most “informative” \( d \) (\( \ll D \)) samples for

- data acquisition (e.g., offline sensor selection)
- data selection (e.g., outlier rejection, censoring)
- Or, combination of above — hybrid
State of the art

- Compressive sensing: sparse signal recovery
  \[\text{Donoho-2006}, \text{Candès-Wakin-2008}\]

- Sensor selection: model-driven
  - convex optimization: design \(\{0, 1\}^M\) selection vector
    \[\text{Joshi-Boyd-2009}, \text{Chepuri-Leus-2015}\]
  - greedy methods and heuristics: submodularity
    \[\text{Krause-Singh-Guestrin-08}, \text{Ranieri-Chebira-Vetterli-14}\]

- Censoring (or data selection) and outlier rejection: data-driven
  \[\text{Rago-Willett-Shalom-96}, \text{Msechu-Giannakis-12}\]

Model-driven, data-driven, or hybrid?
Linear regression setup—model-driven

- Observations follow
  \[ x_m = a_m^T \theta + n_m, \quad m = 1, 2, \ldots, D \]
  - \( \theta \in \mathbb{R}^p \) unknown parameter
  - \( n_m \) i.i.d. zero-mean unit-variance Gaussian noise

Problem statement

Given \( \{a_m\} \) and noise pdf, i.e., only the data model, design \( w \) to select the best subset of \( d (\ll D) \) sensors

- Best subset of \( d (\ll D) \) sensors are obtained by optimizing a scalar function (trace, max. eigenvalue, log det) of the CRB matrix of \( \theta \):
  \[
  f(w) = g \left\{ \left( \sum_{m=1}^{D} w_m a_m a_m^T \right)^{-1} \right\}
  \]
Sparse sensing function can be designed by solving

$$\min_{\mathbf{w} \in \mathcal{W}} f(\mathbf{w})$$

$$\mathcal{W} = \{\mathbf{w} \in \{0, 1\}^D \mid \|\mathbf{w}\|_0 = d\}.$$ 

The above problem is convex on $\mathbf{w}$ if the set $\mathcal{W}$ is relaxed to $\mathcal{W}_c = \{\mathbf{w} \in [0, 1]^D \mid \|\mathbf{w}\|_1 = d\}$.

Monotone submodular cost functions can be optimized using near-optimal greedy methods.

+ Sampler needs to be designed offline only once, and samplers are obtained by optimizing ensemble performance.
- Design might suffer from any possible outliers or model mismatch.
Model-driven design for other inference tasks

- Can be generalized to **nonlinear models** by optimizing scalar functions of the **Cramér-Rao bound matrix**.


- Samplers for nonlinear models with **correlated errors** can be obtained by solving a convex program.


- Samplers can be designed for **detection problems** by optimizing **Kullback-Leibler or Bhattacharyya distance measures**.

Linear regression setup—data-driven

- The output data $\{x_m\}_{m=1}^D$ is possibly contaminated with up to $o$ outliers.
- We know the (uncontaminated) data model

$$\bar{x}_m = a_m^T\theta + n_m, \ m = 1, 2, \ldots, D$$

- $\theta \in \mathbb{R}^p$ unknown parameter
- $n_m$ i.i.d. zero-mean unit-variance Gaussian noise

Problem statement

Given $\{x_m\}$, $\{a_m\}$, and noise pdf:

(a) design $w$ to censor less-informative samples and reject outliers
(b) estimate $\theta$ that performs using the uncensored data
Data-driven design

- Data samples with smaller residuals are informative; more generally the one’s with large likelihood
- Sensing function is obtained by solving

\[
\min_{\mathbf{w} \in \mathcal{W}, \theta} \sum_{m=1}^{D} w_m \left( x_m - a_m^T \theta \right)^2 \iff \min_{\mathbf{w} \in \mathcal{W}} r(\mathbf{w})
\]

\[
r(\mathbf{w}) = x_w^T \left( I - A_w (A_w^T A_w)^{-1} A_w^T \right) x_w.
\]

\[
A_w = \text{diag}_r(\mathbf{w}) A; x_w = \text{diag}_r(\mathbf{w}) x.
\]

- Also known as least trimmed squares.

- This problem is non-convex in general
  - Can be convexified for linear Gaussian case
  - Markov chain Monte Carlo methods (e.g., Metropolis-Hastings sampling)
For linear Gaussian models, the cost function amounts to sparsity based outlier rejection methods

[Fuchs-1999], [Rousseeuw-Leroy-2005], [Giannakis et al.-2011]

Can be generalized to **nonlinear models** by optimizing likelihood function parameterized with \( \theta \) and \( w \).


+ Designs are robust to outliers.
  - Sampler needs to be designed for each data realization, and samplers are obtained by optimizing instantaneous measure (performance could be bad).
Data-driven samplers are robust to outliers, but don’t take the resulting inference performance (i.e., MSE) into account.

Model-driven samplers are MSE optimal, but are not robust to outliers.

Hybrid model-data-driven designs allow to combine (and trade) the above two advantages.

+ Designs are robust to outliers, and take into account the inference performance.
- Sampler needs to be designed for each data realization.
Optimization problem

- Hybrid model-data-driven sensing scheme jointly optimizes the likelihood function (i.e., residual) and the MSE:

\[
\min_{w \in \mathcal{W}} \ r(w) + \lambda f(w)
\]

\[
\mathcal{W} = \{w \in \{0, 1\}^D \mid \|w\|_0 = d\}.
\]

- \(\lambda \to 0(\infty)\) results in the related data (model)-driven scheme

- This problem is non-convex in general
  - Can be convexified for linear Gaussian case
  - Markov chain Monte Carlo methods (e.g., Metropolis-Hastings sampling)
Hybrid model-data-driven sensing design is equivalent to

\[
\min_{\mathbf{w} \in \mathcal{W}, t_1, t_2} t_1 + \lambda t_2 \\
\text{s.t. } r(\mathbf{w}) \leq t_1, \\
\qquad f(\mathbf{w}) \leq t_2.
\]

Using Schur complement and \( \Phi^T \Phi = \text{diag}(\mathbf{w}) \)

\[
\min_{\mathbf{w} \in \mathcal{W}, t_1, t_2} t_1 + \lambda t_2 \\
\text{s.t. } \begin{bmatrix}
\mathbf{A}^T \text{diag}(\mathbf{w}) \mathbf{A} & \mathbf{A}^T \text{diag}(\mathbf{w}) \mathbf{x} \\
\mathbf{x}^T \text{diag}(\mathbf{w}) \mathbf{A} & t_1 - \mathbf{x}^T \text{diag}(\mathbf{w}) \mathbf{x}
\end{bmatrix} \succeq 0, \\
\qquad f(\mathbf{w}) \leq t_2.
\]

The above problem is convex after relaxing \( \mathcal{W} \) to \( \mathcal{W}_c \).
Metropolis-Hastings sampler

- Powerful concept for generating samples from complex distributions:
  1. Each iteration generates a proposal $\tilde{w}$ from some proposal distribution $q(\tilde{w} \mid w^{(j-1)})$.
  2. The new sample $w^{(j)}$ is obtained as

$$w^{(j)} = \begin{cases} \tilde{w} & \text{with probability } \alpha_j \\ w^{(j-1)} & \text{with probability } 1 - \alpha_j \end{cases}$$

$$\alpha_j = \min \left\{ \frac{p_t(\tilde{w}) q(w^{(j-1)} \mid \tilde{w})}{p_t(w^{(j-1)}) q(\tilde{w} \mid w^{(j-1)})}, 1 \right\}$$

$$p_t(w) \propto \exp \left( - \frac{1}{2\sigma^2} (r(w) + \lambda f(w)) \right)$$

- Doesn’t involve convex relaxations ($\mathcal{W}$ to $\mathcal{W}_c$).
Numerical experiments

Best performance of the hybrid scheme is not necessarily in between the best performances of the data-driven and model-driven schemes.
Numerical experiments

Error distribution achieved with the MH method almost coincides with that of exhaustive search.
Conclusions and future directions

- Model-driven, data-driven, hybrid sparse sensing
  - for basic inference problems
  - respective strengths and weaknesses

Future directions

- Correlated observations, clustering, and classification
- Greedy algorithms (submodular)

Thank You!!