Game Theory in the Talmud: Bankruptcy Problem

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Am I getting my money's worth?

- Setup: users who pay differently for some shared resource(s) and expect allocations accordingly.
- If resources are aplenty: allocation should not be an issue.
- If resources are scarce: allocation can be challeging.
- We need to answer whether allocation is fair i.e., is every user getting his money's worth subject to the resource constraint.
- We look at examples of allocation schemes in the Jewish text (Talmud) and the rationale behind them.
- We consider particularly, the CG-consistent allocation scheme, study its properties and its relationship to coalitional game theoretic concepts.

The bankruptcy problem $(E; d)$

[Aumann, Maschler] Bankruptcy problem in the Babylonian Talmud.

- An estate has to be divided among three creditors $\{1, 2, 3\}$.
- The estate's worth E is less than the sum of the claims $d_1 + d_2 + d_3$.
- The Mishna (law) suggests the following allocation:

Estate	100	200	300
100	$33\frac{1}{3}$	$33\frac{1}{3}$	$33\frac{1}{3}$
200	50	75	75
300	50	100	150

- Is there a general rule behind the above allocation?
- What is the principle that guides such a rule (if it exists)?

The contested garment principle

- "Two claim a garment, one claims its all, the other half, then the one is awarded $\frac{3}{4}$ and the other $\frac{1}{4}$".
- Consistent Garment principle: Given bankruptcy problem $(E;(d_1,d_2))$, the CG principle suggests the following allocation:

$$x_i = \frac{E - (E-d_1)^+ - (E-d_2)^+}{2} + (E-d_j)^+. \tag{1}$$

- Instead of an awarding an allocation, we can think of the rule as dividing losses as follows:

$$x_i = \min\{E, d_i\} - \frac{\min\{E,d_1\} + \min\{E,d_2\} - E}{2}. \tag{2}$$

- CG-consistent allocation is monotonic in E (for fixed d).

CG-consistent solution

Bankruptcy Problem

A bankruptcy problem is a pair $(E; d)$ with $d = (d_1, d_2, \cdots, d_n)$ s.t., $0 \le d_1 \le d_2 \le \cdots \le d_n$ and $0 \le E \le (d_1 + d_2 + \cdots + d_n)$.

A solution to such a problem is $x = (x_1, x_2, \cdots, x_n)$ s.t., $x_1 + x_2 + \cdots + x_n = E$.

CG-consistent solution

A solution x to $(E; d)$ is called CG-consistent if $\forall i, j$, division of $x_i + x_j$ prescribed by the CG principle for claims (d_i, d_j) is (x_i, x_j).

CG-consistent solution

Theorem: Existence and Uniqueness of consistent solution

Every bankruptcy problem has a unique consistent solution.

- Uniqueness proof: By contradiction.
- Existence proof: Specify an allocation rule and prove CG-consistency.
- If $E \leq \frac{D}{2}$, use rule1(E, d, n)
  - if $E \leq \frac{nd_1}{2}$: set allocation $x_i = \frac{E}{n}$, $\forall i$.
  - else: allocate $\frac{d_1}{2}$ to 1 and use rule1$(E - \frac{d_1}{2}, d_{-1}, n-1)$.
- else, use rule2(E, d, n)
  - if $(D - E) \leq \frac{nd_1}{2}$: set allocation $x = d - \frac{D-E}{n}$.
  - else: allocate $\frac{d_1}{2}$ to 1 and use rule2$(E - \frac{d_1}{2}, d_{-1}, n-1)$.

CG-consistent allocation

- For rule1, we have allocation of the form $\frac{d_1}{2}, \cdots, \frac{d_k}{2}, \frac{E-\sum_1^k \frac{d_i}{2}}{n-k}, \cdots, \frac{E-\sum_1^k \frac{d_i}{2}}{n-k}$.
- For rule2, we have allocations of the form $\frac{d_1}{2}, \cdots, \frac{d_k}{2}, d_{k+1} - \frac{\sum_{k+1}^n d_i-(E-\sum_1^k \frac{d_i}{2})}{n-k}, \cdots, d_n - \frac{\sum_{k+1}^n d_i-(E-\sum_1^k \frac{d_i}{2})}{n-k}$.
- We can show that the allocation scheme described above is consistent.
- This would complete the proof of existence of the consistent solution to the bankruptcy problem.
- Note: rule2 is like rule1 but it divides the losses among the creditors.

Self-consistent allocation rules

Self-consistent rule

A rule f is called self-consistent if

$$f(E;d) = x \text{ implies that } f(x(S);d|S) = x|S \text{ for all S.} \qquad (3)$$

Corollary

The CG-consistent rule is self-consistent.

“Proportional to claim“ and “constrained equal division“ rules are also self-consistent.

Dual of an allocation rule

Dual allocation rules

The dual f^* of a rule f is given by

$$f^*(E; d) = d - f(D - E; d) \tag{4}$$

Self-dual rule

A self-dual rule is an allocation rule such that $f^* = f$.

A self-dual rule treats losses and awards in the same way.

Constrained equal award [Maimonides]

The CEA solution of a bankruptcy problem $(E; d)$ is of the form $(a \wedge d_1, a \wedge d_2, \cdots, a \wedge d_n)$.

Theorem

Each bankruptcy problem has a unique CEA solution.

Proof: $\sum x_i = \sum_i (a \wedge d_i) := f(a)$ is continuous strictly increasing function $f : [0, d_n] \to [0, D]$. Hence f attains every point of $[0, D]$ precisely once.

- Rabad's allocation rule [RAR]:
    - When $E \leq d_1$, allocate equally.
    - When $d_1 < E \leq d_2$, allocate $\frac{d_1}{n}$ to 1. The remaining is allocated equally among the remaining $n - 1$ creditors.
    - and so on.
    - The rule is not defined beyond d_n.

Dual rules: dealing with losses

- We have n bidders with bids $b_1 < b_2 < \cdots < b_n$, if bidder n drops out $n-1$ wins.
- The seller sees this as a loss and demands a payment of $b_n - b_{n-1}$ from n.
- How does the seller get payments if everyone drops out?
- CEA rule: Divide losses equally among all bidders subject to no one paying more than his bid.
- RAR rule: Divide losses into increments $b_1, b_2 - b_1, b_3 - b_2, \cdots, b_n - b_{n-1}$. The first increment is shared by everyone equally, the second by everyone except 1 equally and the last one by only n.

Self-Dual rules

- The creditors in the bankruptcy might see the transaction as a loss rather than an award.
- Instead of looking at the bankruptcy solution as a award granting rule, we might see it as a loss allocation rule.
- We wish that the allocation rules treat losses and awards equally.
- Self-dual rules are allocation rules that treat losses and awards equally.
- CG-consistent allocation rule is a self-dual rule.

The significance of the halfway mark in the CG-consistent solution

- We see that the CG-consistent rule has a qualitative change at the halfway mark.
- This is based on the Talmudic principle that 'More than half is like the whole'.
- The CG-consistent rule are based on the above principle and following psychological presumptions.
- When worth of debtor's property is more than half, the lender relies on it as a guarantee.
- When it is less than half, the lender gave the loan on trust and hence he is not justifying in taking away the property to recover the loan.
- In the CG-consistent rule when $E \geq \frac{D}{2}$, the **losses** are divided equally.
- When $E \leq \frac{D}{2}$, the **awards** are divided equally.
- When a lender i receives $\frac{d_i}{2}$, he is willing to write off the debt as paid.

Relating CG-consistent solution to CEA solution

Theorem

The consistent rule is the unique self-dual rule that when $E \leq \frac{D}{2}$ assigns to $(E; d)$ the constrained equal award solution of $(E; \frac{d}{2})$.

Properties of the CG-consistent solution

- An allocation rule f is **monotonic** is $f_i(E; d)$ is non decreasing function of E for all fixed i and d.
- The solution x of a bankruptcy problem $(E; d)$ is **order preserving** if $0 \leq x_1 \leq x_2 \leq \cdots \leq x_n$ and $0 \leq d_1 - x_1 \leq d_2 - x_2 \leq \cdots \leq d_n - x_n$.

The CG-consistent solution, division in proportion to claims, CEA rule and its dual, the RAR rule (as long as it is defined) are monotonic and order preserving.

Coalitional Procedure

Consider the following three rules of division

1. Divide E between $\{1\}$ and $\{2, \cdots, n\}$ in accordance with CG-consistent solution of $(E; (d_1, d_2 + \cdots + d_n))$.
2. Assign equal awards to all creditors.
3. Assign equal losses to all creditors.

We apply

- 1 if $\frac{nd_1}{2} \leq E \leq D - \frac{nd_1}{2}$.
- 2 if $E \leq \frac{nd_1}{2}$.
- 3 if $D - \frac{nd_1}{2} \leq E$.

The above procedure is called the coalitional procedure.

Coalitional Procedure

Theorem

The coalitional procedure yields the consistent solution to all bankruptcy problems.

A coalitional game formulation

- We denote the coalitional game as (N, v) where v associates payoff to all subsets of N.
- Define $(N, v_{(E;d)})$ as the bankruptcy game with payoff rule as follows

$$v_{E;d}(S) = (E - d(N - S))^{+} \tag{5}$$

Theorem

The consistent solution of a bankruptcy problem $(E; d)$ is the nucleolus of the corresponding bankruptcy game $(N, v_{(E;d)})$.

Standard Solution and CG-consistent solution

The standard solution of a 2-person game v with player set $\{1,2\}$ is given by

$$x_i = \frac{v(1,2) - v(1) - v(2)}{2} + v(i). \tag{6}$$

- The nucleolus, kernel, pre-kernel and Shapley value of a 2-person game all coincide with the standard solution.
- Lemma: The CG-consistent solution of 2-person bankruptcy game is the standard solution of the corresponsing game.

The kernel of the bankruptcy game $v_{E;d}$ consists of a single point, namely the consistent solution of the problem $(E; d)$.

Thank You

Thank You!

Notions of fairness: An example from Networks

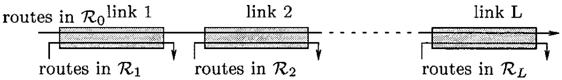

Figure: The Linear Network

	Max-min	Min Potential delay	Proportional	Max-rate
λ_0	$\frac{1}{2}$	$\frac{1}{\sqrt{L}+1}$	$\frac{1}{L+1}$	0
λ_1	$\frac{1}{2}$	$\frac{\sqrt{L}}{\sqrt{L}+1}$	$\frac{L}{L+1}$	1
Total	$\frac{L+1}{2}$	$L+1-\sqrt{L}$	$\frac{L^2+1}{L+1}$	L

- Different allocation schemes motivated by different objectives.
- We see that max-min fair allocations are 'equitable' but inefficient whereas max-rate allocations are efficient but unfair.