# Power Minimization for CDMA under Colored Noise

Arun Padakandla and Rajesh Sundaresan, *Senior Member, IEEE*

***Abstract*—Rate-constrained power minimization (PMIN) over a code division multiple-access (CDMA) channel with correlated noise is studied. PMIN is shown to be an instance of a separable convex optimization problem subject to linear ascending constraints. PMIN is further reduced to a dual problem of sum-rate maximization (RMAX). The results highlight the underlying unity between PMIN, RMAX, and a problem closely related to PMIN but with linear receiver constraints. Subsequently, conceptually simple sequence design algorithms are proposed to explicitly identify an assignment of sequences and powers that solve PMIN. The algorithms yield an upper bound of $2N-1$ on the number of distinct sequences where $N$ is the processing gain. The sequences generated using the proposed algorithms are in general real-valued. If a rate-splitting and multi-dimensional CDMA approach is allowed, the upper bound reduces to $N$ distinct sequences, in which case the sequences can form an orthogonal set and be binary $\pm1$-valued.**

***Index Terms*—Code division multiple access (CDMA), separable convex optimization, linear receivers, inverse eigenvalue problems, sequences.**

## I. Introduction

We consider the code division multiple-access (CDMA) channel

$$Y = \sum_{k=1}^{K} s_k X_k + Z. \tag{1}$$

There are $K$ users in the system. User $k$ modulates the vector (sequence) $s_k \in \mathbb{R}^N$ by its data symbol $X_k$ and transmits $s_k X_k$ over the air. The received signal $Y$ is the superposition of the transmissions embedded in Gaussian noise $Z$. The random vector $Z$ is jointly Gaussian with mean zero and a positive definite (pd) covariance $\Sigma$. Equation (1) models the received CDMA signal at the output of a channel with a frequency-flat response. CDMA may be employed on such a system to get some resilience to jamming. We assume symbols are synchronous and study the following problem:

**PMIN**: User $k$ demands reliable transmission at a minimum rate $r_k$ bits/symbol. The goal is to assign sequences and powers to users so that despite their mutual interference and noise, each of the users can transmit reliably at or greater than their demanded rates, and the sum of the received powers (energy/symbol) at the base-station is minimized.

The above problem is one of intra-cell interference management for efficient use of system resources (power) in an overloaded CDMA system ($K \geq N$). Our main contributions are (1) a solution to PMIN for any pd $\Sigma$; (2) a new finite-step algorithm for optimal sequence assignment. Our solution provides theoretical limits on rate-constrained sum power allocation and aims to manage intra-cell interference. The motivations for minimization of sum received power are as follows. First, it enables users to meet their requirements with lesser overall transmitted power, when viewed from a system perspective. Second, this sum received power is a scaled version of the interference seen at a neighboring receiver, when the neighbor models the users' locations as identically distributed over the cell.

PMIN was solved for the case $\Sigma = I_N$ by Guess [1]. A dual problem of sum rate maximization (RMAX) given $K$ users, processing gain $N$, and power constraints $p_1, p_2, \cdots, p_K$ was solved for any pd $\Sigma$ by Viswanath & Anantharam [2]. (See Viswanath & Anantharam [3] for the $\Sigma = I_N$ case). They also address a version of PMIN, say PMIN-LR, where the receivers are constrained to be linear. Our solution extends Guess' result in [1] to a system with any pd $\Sigma$. We remark that we consider a system where the sequences are real-valued and not restricted to a finite alphabet set. Later in the paper we remark how we may restrict attention to orthogonal sequences alone. Such sequences can then come from a finite alphabet set.

Our method to solve PMIN is interesting in itself. It brings to light a unity among PMIN, RMAX, and PMIN-LR. They are instances of a class of separable convex optimization problems with linear ascending inequality and equality constraints [4]. Furthermore, we prove a strong duality between PMIN and RMAX : an instance of PMIN can be mapped to an instance of RMAX and vice versa. This implies that a solution to one can be mapped to a solution on the other.

Several algorithms to identify optimal sequence sets under frequency-selective fading, asynchronism, and under requirements of scalability ([3], [5], [6], [7], [8], and [9]) are known. Viswanath & Anantharam [3] provide recursive algorithms based on $T$-transforms to identify sequences that solve RMAX. Tropp et al. [6] recognize sequence design problems as inverse singular value problems and provide $O(KN)$ finite step algorithms. Anigstein & Anantharam [9] provide an iterative algorithm to identify sequence sets that

Paper approved by L. K. Rasmussen, the Editor for Iterative Detection Decoding and ARQ of the IEEE Communications Society. Manuscript received April 3, 2008; revised August 6, 2008.

A. Padakandla is with the Department of Electrical Engineering and Computer Science, University of Michigan, Ann Arbor 48109-2122, USA (e-mail: arun.padakandla@gmail.com).

R. Sundaresan is with the Department of Electrical Communication Engineering, Indian Institute of Science, Bangalore 560012, India (e-mail: rajeshs@ece.iisc.ernet.in).

This work was supported by the Department of Science and Technology under Grant DST0748 and by the University Grants Commission under Grant Part (2B) UGC-CAS-(Ph.IV).

Material in this paper was published in part at the International Symposium on Information Theory held in Nice, France, June 2007.

Digital Object Identifier 10.1109/TCOMM.2009.10.080165

achieve minimum total squared correlation. These are suited for distributed implementation. All the algorithms in the above papers (as do the algorithms presented in this one) output sequences in $\mathbb{R}^N$. In contrast, the works of [10], [11], [12], [13], and [14] focus on binary-valued sequence sets. While we do not place such restrictions in the first part of the paper, we do remark in Section III-C that the rate-splitting technique of Guess [1] and Urbanke & Rimoldi [15] enables us to restrict attention to $N$ orthogonal sequences which can further be restricted to come from a finite alphabet. All works described above with the exception of [2] focus on $\Sigma = I_N$.

Our new algorithm (the second contribution) is based on a solution to an inverse eigenvalue problem for rank-one perturbations [16]. The solution can be adapted to provide a new algorithm for RMAX. The distinguishing feature of our algorithm is a bound on the size of the optimal sequence set: at most $2N-1$ sequences are output, a bound independent of the number of users. In a system with a large number of users $K \gg 2N-1$, the upper bound implies savings in downlink signaling for sequence allocation. The algorithm also admits a simple "water-filling" interpretation.

Our paper is organized as follows. We present the solution to PMIN in two steps. In Section II we reduce PMIN to an instance of the aforementioned convex optimization problem. This enables us characterize the minimum sum power and identify necessary and sufficient conditions satisfied by an assignment of sequences and powers that solves PMIN. In Section III we provide finite step algorithms to explicitly identify such an assignment. We also present bounds on the size of the optimal sequence sets, with and without multi-dimensional signaling.

## II. Minimum Sum Power

The search for sequences and powers that solve PMIN may be restricted to a so-called set of tight commuting designs (Definition 1(e)). We prove this in Section II-B and provide a characterization of such designs in Section II-C. In Section II-D we prove PMIN is an instance of a structured convex optimization problem and illustrate the RMAX-PMIN duality. We begin with a precise formulation of the problem.

### A. Problem Statement

The following are some remarks on notation. For integers $i$, $j$, define the integer interval

$$[\![i,j]\!] = \begin{cases} \{i, i+1, \cdots, j\} & \text{if } i \le j \\ \emptyset & \text{otherwise.} \end{cases}$$

For vectors $a, b \in \mathbb{R}^n$, define $a \cdot b$, $a/b$, $\exp\{a\}$, and $\log a \in \mathbb{R}^n$ as follows[1]: $(a \cdot b)_i = a_i b_i$, $(a/b)_i = a_i / b_i$ (when $b_i \neq 0$), $(\exp\{a\})_i = \exp\{a_i\}$, and $(\log a)_i = \log a_i$ (when $a_i > 0$) for $i \in [\![1, n]\!]$. The quantity $\|a\|$ denotes the Euclidean norm of $a$, and $\mathrm{diag}(a)$ the diagonal $n \times n$ matrix with $i$th diagonal entry $a_i$. Ordering of components plays a key role in this paper. We denote by $a_{[i]}$ the $i$th largest component of $a$.

[1] The log and exp functions may be taken with base 2 in which case the unit of rate is bits/symbol.

$|A|$ is the determinant of a square $\mathbb{R}$-valued matrix $A$. We say $A$ *is pd* if the matrix $A$ is positive definite (i.e., symmetric with strictly positive eigenvalues). $\mathrm{tr}(A)$ is the trace of the square matrix $A$. For a symmetric $A$, $\mathrm{eig}(A)$ denotes the eigenvalues of $A$ in decreasing order. (In this paper, we deal with eig of only pd matrices). For a pd $A$ we denote its square root and inverse square root by $A^{\frac{1}{2}}$ and $A^{-\frac{1}{2}}$, respectively.

There are $K$ users in the system. The $k$th user is received with power at most $p_k$, where $p_k$ naturally depends on the actual transmit power constraint and the distance of the user $k$ from the uplink receiver. The $k$th user transmits over an $N$-dimensional space using a unit-norm signature vector $s_k \in \mathbb{R}^N$ satisfying $s_k^t s_k = 1$. The signature matrix $S := [s_1\ s_2\ \cdots\ s_K]$ is of size $N \times K$ with columns $s_k, k \in [\![1, K]\!]$. The received power matrix is $P := \mathrm{diag}(p_1, \cdots, p_K)$. The sum received power is $\mathrm{tr}(P) = \sum_{k=1}^K p_k$. For a subset of users $J \subseteq [\![1, K]\!]$, let $S_J := [s_j]_{j \in J}$ denote the submatrix of $S$ of size $N \times |J|$ obtained by keeping only those columns whose indices are in $J$. Let $P_J := \mathrm{diag}\left((p_k)_{k \in J}\right)$.

The set of all sequence matrices $\mathcal{S}$ is the set of all $N \times K$ matrices whose columns have unit norm, i.e.,

$$\mathcal{S} := \left\{ S \in \mathbb{R}^{N \times K} : \left(S^t S\right)_{ii} = 1 \text{ for } i \in [\![1, K]\!] \right\}.$$

The set of all power matrices is

$$\mathcal{P} := \left\{ P \in \mathbb{R}_+^{K \times K} : P = \mathrm{diag}(p) \text{ for some } p \in \mathbb{R}_+^K \right\}.$$

For a fixed $(S, P) \in \mathcal{S} \times \mathcal{P}$ and a fixed pd covariance matrix $\Sigma$, the capacity region for the Gaussian MAC in (1) is

$$C(S, P, \Sigma) = \bigcap_{J \subseteq [\![1,K]\!]} \left\{ r \in \mathbb{R}_+^K : \sum_{k \in J} r_k \le \frac{1}{2} \log \left| I_N + \Sigma^{-1} S_J P_J S_J^t \right| \right\}. \quad (2)$$

Here $r$ denotes a rate vector whose $k$th component $r_k$ is the rate achieved by user $k$. $SPS^t$ is the signal covariance matrix and $\Sigma + SPS^t$ is the covariance matrix of the channel output.

*Definition 1:* Fix $r \in \mathbb{R}_+^K$ and a pd $\Sigma$ of size $N \times N$. We use the following terminology.

(a) $(S, P)$ is a *design* if and only if (iff) $(S, P) \in \mathcal{S} \times \mathcal{P}$.
(b) $(S, P)$ is a *design for $r$ on $\Sigma$* iff $(S, P)$ is a design and $r \in C(S, P, \Sigma)$.
(c) $(S, P)$ is a *tight design for $r$ on $\Sigma$* iff $(S, P)$ is a design for $r$ on $\Sigma$ and $\sum_{k=1}^K r_k = \frac{1}{2} \log |I_N + \Sigma^{-1} SPS^t|$, i.e., the upper bound in (2) is achieved when $J = [\![1, K]\!]$.
(d) $(S, P)$ is a *commuting design for $r$ on $\Sigma$* iff $(S, P)$ is a design for $r$ on $\Sigma$, and $SPS^t$ and $\Sigma$ commute.
(e) $(S, P)$ is a *tight commuting design for $r$ on $\Sigma$* iff $(S, P)$ is both a tight design and a commuting design for $r$ on $\Sigma$.

We now reformulate PMIN.

**PMIN** : Given $r \in \mathbb{R}_+^K$ and a pd $\Sigma$ of size $N \times N$, find

$$P_{\min} = \min \left\{ \mathrm{tr}(P) : (S, P) \text{ is a design for } r \text{ on } \Sigma \right\}.$$

□

Using the facts that $P$ is diagonal, the diagonal entries of $S^t S$ are 1, and $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ when the matrix products $AB$

and $BA$ are both well-defined, we see that

$$\begin{aligned} \text{tr}(P) &= \text{tr}\left(PS^tS\right) = \text{tr}\left(SPS^t\right) \\ &= \text{tr}\left(\Sigma + SPS^t\right) - \text{tr}(\Sigma). \end{aligned} \tag{3}$$

Since tr$(\Sigma)$ is a constant, PMIN is solved if we can identify the vector with the minimum sum of components in the set

$$\{\text{eig}(\Sigma + SPS^t) : (S, P) \text{ is a design for } r \text{ on } \Sigma\}. \tag{4}$$

### *B. Sufficiency of tight commuting designs*

The following lemma provides simple and useful observations on orthonormal transformations.

*Lemma 2:* Let $Q$ be a given $N \times N$ orthonormal matrix. Let $(\bar{S}, \bar{P})$ be a design and let $U$ be an orthonormal matrix that diagonalizes $\bar{S}\bar{P}\bar{S}^t$. Define $\hat{S} := QU^t\bar{S}$. The following are true.

(i) $\hat{S} \in \mathcal{S}$, and therefore $(\hat{S}, \bar{P})$ is a design.
(ii) $Q$ diagonalizes $\hat{S}\bar{P}\hat{S}^t$, and $Q^t\hat{S}\bar{P}\hat{S}^tQ = U^t\bar{S}\bar{P}\bar{S}^tU$. In particular, eig$(\hat{S}\bar{P}\hat{S}^t) =$ eig$(\bar{S}\bar{P}\bar{S}^t)$.
(iii) $C\left(\bar{S}, \bar{P}, I_N\right) = C\left(\hat{S}, \bar{P}, I_N\right)$, i.e., the capacity region is unaffected.

*Proof:* This is obvious and hence omitted. ■

We now map a design on $I_N$ to a design on a pd $\Sigma$ and vice versa. Given a pd $\Sigma$, define the function $f_\Sigma : \mathcal{S} \times \mathcal{P} \to \mathcal{S} \times \mathcal{P}$ in the following fashion. For each $(S, P)$, set $A := \Sigma^{-\frac{1}{2}}SP^{\frac{1}{2}}$ and let $(S, P) \overset{f_\Sigma}{\mapsto} (\bar{S}, \bar{P})$ where

$$\begin{aligned} \bar{P} &:= \text{diag}\left(\|a_1\|^2,\ \|a_2\|^2,\ \cdots,\ \|a_K\|^2\right) \\ \bar{S} &:= A\bar{P}^{-\frac{1}{2}} \end{aligned}$$

with $a_k$ the columns of $A$. Clearly, $\Sigma^{-\frac{1}{2}}SP^{\frac{1}{2}} = \bar{S}\bar{P}^{\frac{1}{2}}$ and therefore

$$\bar{S}\bar{P}\bar{S}^t = \Sigma^{-\frac{1}{2}}SPS^t\Sigma^{-\frac{1}{2}}. \tag{5}$$

It is also straightforward to verify that $(\bar{S}, \bar{P}) = f_\Sigma(S, P)$ is a design, and that $f_\Sigma$ is invertible with inverse $f_{\Sigma^{-1}}$.

*Lemma 3:* For a pd $\Sigma$, the design $(S, P)$ on $\Sigma$ and the design $f_\Sigma(S, P)$ on $I_N$ have identical capacity regions, i.e., $C(S, P, \Sigma) = C\left(f_\Sigma(S, P), I_N\right)$.

*Proof:* Let $(\bar{S}, \bar{P}) = f_\Sigma(S, P)$. Fix arbitrary $J \subseteq [\![1, K]\!]$. We first observe that by the diagonal nature of $P$ and $\bar{P}$, we have

$$\begin{aligned} \Sigma^{-\frac{1}{2}}S_JP_J^{\frac{1}{2}} &= \Sigma^{-\frac{1}{2}}\left(SP^{\frac{1}{2}}\right)_J = \left(\Sigma^{-\frac{1}{2}}SP^{\frac{1}{2}}\right)_J \\ &= \left(\bar{S}\bar{P}^{\frac{1}{2}}\right)_J = \bar{S}_J\bar{P}_J^{\frac{1}{2}}. \end{aligned} \tag{6}$$

This can be verified, for example, by setting $e_j$ to be the unit vector with 1 in the $j$th location and 0 elsewhere, $e_J := [e_j]_{j\in J}$, the matrix whose columns are unit vectors $e_j$ with $j \in J$. Then $S_J = Se_J$, $P_J = e_J^tPe_J$, $e_J^te_J = I_{|J|}$, and $e_Je_J^t$ is a diagonal matrix with 1 in positions $(j, j), j \in J$, and 0 elsewhere. Clearly then $S_JP_J^{\frac{1}{2}} = (Se_J)(e_J^tP^{\frac{1}{2}}e_J) = (S)(e_Je_J^tP^{\frac{1}{2}})(e_J) = SP^{\frac{1}{2}}e_J = (SP^{\frac{1}{2}})_J$. Similarly $\bar{S}_J\bar{P}_J^{\frac{1}{2}} = (\bar{S}\bar{P}^{\frac{1}{2}})_J$, and (6) follows. Consequently,

$$\begin{aligned} \left|I_N + \Sigma^{-1}S_JP_JS_J^t\right| &= \left|I_N + \Sigma^{-\frac{1}{2}}S_JP_JS_J^t\Sigma^{-\frac{1}{2}}\right| \\ &= \left|I_N + \bar{S}_J\bar{P}_J\bar{S}_J^t\right|. \end{aligned}$$

Since $J$ was arbitrary, the equality of the capacity regions now follows from (2). ■

*Remark 1:* Thus, $(S, P)$ is a design (respectively tight design) for $r$ on $\Sigma$ iff $(\bar{S}, \bar{P}) = f_\Sigma(S, P)$ is a design (respectively tight design) for $r$ on $I_N$.

*Remark 2:* If $(\bar{S}, \bar{P})$ is a design for $r$ on $I_N$ and $\bar{S}\bar{P}\bar{S}^t$ commutes with $\Sigma$, then $(S, P) = f_{\Sigma^{-1}}(\bar{S}, \bar{P})$ is a commuting design for $r$ on $\Sigma$. This is because (5) applied to $f_{\Sigma^{-1}}$ yields $SPS^t = \left(\Sigma^{\frac{1}{2}}\right)\left(\bar{S}\bar{P}\bar{S}^t\right)\left(\Sigma^{\frac{1}{2}}\right)$ and the parenthesized matrices on the right side commute. Moreover, if $U$ diagonalizes $\Sigma$, i.e., $U^t\Sigma U = \Lambda_\Sigma$, then $U$ diagonalizes both $\bar{S}\bar{P}\bar{S}^t$ and $SPS^t$ into $\Lambda_{\bar{S}\bar{P}\bar{S}^t}$ and $\Lambda_{SPS^t}$, respectively, with the additional property

$$\Lambda_{SPS^t} = \Lambda_{\bar{S}\bar{P}\bar{S}^t} \cdot \Lambda_\Sigma. \tag{7}$$

As for $\Sigma = I_N$ (Guess [1, Lem. 1]), we only need to focus on tight designs.

*Lemma 4:* Fix $r \in \mathbb{R}_+^K$ and a pd $\Sigma$. If $(S, P)$ is a design for $r$ on $\Sigma$, there exists a tight design $(\tilde{S}, \tilde{P})$ for $r$ on $\Sigma$ with tr$(\tilde{P}) \leq$ tr$(P)$.

*Proof:* The proof of [1, Lem. 1] uses continuity of the mapping $P \mapsto C(S, P, \Sigma)$ in $P$, and extends verbatim to any pd $\Sigma$. ■

Furthermore, it is sufficient to restrict our search to tight commuting designs:

*Theorem 5:* If $(S, P)$ is a tight design for $r$ on $\Sigma$, there exists a tight commuting design $(\tilde{S}, \tilde{P})$ for $r$ on $\Sigma$ that satisfies tr$(\tilde{P}) \leq$ tr$(P)$.

*Proof:* Let the orthonormal matrix $Q$ diagonalize $\Sigma$ in the *increasing* order of eigenvalues, i.e.,

$$Q^t\Sigma Q = \Lambda_\Sigma = \text{ diag}\left(\sigma_1^2, \cdots, \sigma_N^2\right), \tag{8}$$

where $\sigma_n^2 \leq \sigma_{n+1}^2$ for $n \in [\![1, N-1]\!]$. Let $(\bar{S}, \bar{P}) = f_\Sigma(S, P)$ and let the orthonormal matrix $U$ diagonalize $\bar{S}\bar{P}\bar{S}^t$ in *decreasing* order. Consider the following design transformations:

$$\begin{array}{ccccccc} (S, P) & \overset{f_\Sigma}{\longmapsto} & (\bar{S}, \bar{P}) & \longmapsto & (\hat{S}, \bar{P}) & \overset{f_{\Sigma^{-1}}}{\longmapsto} & (\tilde{S}, \tilde{P}) \\ \Sigma & & I_N & & I_N & & \Sigma \end{array}$$

where $(S, P)$ and $(\tilde{S}, \tilde{P})$ are designs on $\Sigma$, $(\bar{S}, \bar{P})$ and $(\hat{S}, \bar{P})$ are designs on $I_N$, as indicated in the second row above, and $\hat{S} := QU^t\bar{S}$ as in Lemma 2. Then Lemmas 2 and 3 assure us that the capacity regions are preserved across the above transformations, and so all these designs are tight on the respective channels. Also from Lemma 2(ii), $\hat{S}\bar{P}\hat{S}^t$ and $\Sigma$ are both diagonalized by $Q$ and therefore commute. Remark 2 then shows that $(\tilde{S}, \tilde{P})$ is a commuting design for $r$ on $\Sigma$.

We now show that tr$(\tilde{P}) \leq$ tr$(P)$. From the definition of $f_\Sigma$, observe that

$$\text{tr}(P) = \text{tr}(SPS^t) = \text{tr}(\Sigma^{\frac{1}{2}}\bar{S}\bar{P}\bar{S}^t\Sigma^{\frac{1}{2}}) = \text{tr}(\Sigma\bar{S}\bar{P}\bar{S}^t).$$

Lemma 2(ii) shows $\Lambda_{\bar{S}\bar{P}\bar{S}^t} = \Lambda_{\hat{S}\bar{P}\hat{S}^t}$ with diagonalizing matrices $U$ and $Q$, respectively. To complete the proof we use a result from Marshall & Olkin [17, Lemma 9.H.1.h]: for any pair of $N \times N$ pd matrices $A$ and $B$,

$$\text{tr}(AB) \geq \sum_{i=1}^{N} \text{eig}(A)_{N-i+1}\text{eig}(B)_i, \tag{9}$$

where $\text{eig}(A)_i$ is the $i$th component of the ordered set of eigenvalues of $A$. Using these facts, we get

$$\begin{aligned}\text{tr}(P) = \text{tr}\left(\Sigma \bar{S}\bar{P}\bar{S}^t\right) &\geq \sum_{i=1}^{N} \text{eig}(\Sigma)_{N-i+1}\text{eig}(\bar{S}\bar{P}\bar{S}^t)_i \\ &= \sum_{i=1}^{N} \text{eig}(\Sigma)_{N-i+1}\text{eig}(\hat{S}\bar{P}\hat{S}^t)_i \\ &\qquad (\text{since } \Lambda_{\hat{S}\bar{P}\hat{S}^t} = \Lambda_{\bar{S}\bar{P}\bar{S}^t}) \\ &= \text{tr}\left(\Lambda_\Sigma \Lambda_{\hat{S}\bar{P}\hat{S}^t}\right) \qquad (10) \\ &= \text{tr}\left(\Lambda_{\tilde{S}\tilde{P}\tilde{S}^t}\right) \qquad (11) \\ &= \text{tr}\left(\tilde{S}\tilde{P}\tilde{S}^t\right) = \text{tr}(\tilde{P})\end{aligned}$$

where (10) follows because $Q$ diagonalizes $\Sigma$ and $\hat{S}\bar{P}\hat{S}^t$ in increasing and decreasing orders, respectively, and (11) is obtained by applying (7) to $\left(\tilde{S},\tilde{P}\right) = f_{\Sigma^{-1}}\left(\hat{S},\hat{P}\right)$. This completes the proof. ■

### C. Characterization of Tight Commuting Designs

To solve PMIN, we saw in Section II-B that it is sufficient to focus on tight commuting designs, and in particular on

$$\Big\{\text{eig}(\Sigma + \tilde{S}\tilde{P}\tilde{S}^t) : (\tilde{S},\tilde{P}) \text{ is a tight commuting design for } r \text{ on } \Sigma\Big\}. \quad (12)$$

Guess [1, Th. 1] characterized this set when $\Sigma = I_N$ as follows. Observe that when $\Sigma = I_N$, all designs are commuting designs. One could then simply focus on $\text{eig}(SPS^t)$.

*Lemma 6:* (Guess [1, Th. 1]) For $r \in \mathbb{R}_+^K$, let

$$\mathbb{L}_I(r) := \left\{ \lambda \in \mathbb{R}_+^N : \begin{array}{c} \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N, \\ \prod_{i=1}^{n}(\lambda_i+1) \geq \prod_{i=1}^{n}\exp\{2r_{[i]}\} \\ \text{for } n \in [\![1, N-1]\!], \\ \prod_{i=1}^{N}(\lambda_i+1) = \prod_{i=1}^{K}\exp\{2r_{[i]}\} \end{array} \right\}.$$

Then

$$\mathbb{L}_I(r) = \left\{\text{eig}(SPS^t) : (S,P) \text{ is a tight design for } r \text{ on } I_N\right\}.$$

Based on the connection between designs on $\Sigma$ and on $I_N$ given in Lemma 3, we may anticipate that a characterization of (12) is related to $\mathbb{L}_I$. This is indeed the case.

*Theorem 7:* Let $r \in \mathbb{R}_+^K$. For a pd $\Sigma$, let $Q$ diagonalize $\Sigma$ in the increasing order of eigenvalues as in (8). Let $\sigma^2 = \left(\sigma_1^2, \sigma_2^2, \cdots, \sigma_N^2\right)$, and let

$$\mathbb{L}_C(r,\sigma^2) := \left\{ \mu \in \mathbb{R}_+^N : \begin{array}{c} \left(\frac{\mu_i}{\sigma_i^2}\right) \geq 1,\ i \in [\![1, N]\!], \\ \prod_{i=1}^{n}\left(\frac{\mu}{\sigma^2}\right)_{[i]} \geq \prod_{i=1}^{n}\exp\{2r_{[i]}\} \\ \text{for } n \in [\![1, N-1]\!], \\ \prod_{i=1}^{N}\left(\frac{\mu}{\sigma^2}\right)_{[i]} = \prod_{i=1}^{K}\exp\{2r_{[i]}\} \end{array} \right\}.$$

Then

$$\mathbb{L}_C(r,\sigma^2) = \left\{ \sigma^2 + l \in \mathbb{R}_+^N : \begin{array}{l} (S,P) \text{ is a tight} \\ \text{commuting design} \\ \text{for } r \text{ on } \Sigma, \text{ and} \\ Q^t SPS^t Q = \text{diag}(l) \end{array} \right\}.$$

*Proof:* Let $(S,P)$ be a tight commuting design for $r$ on $\Sigma$ so that $SPS^t$ is diagonalized by $Q$. Define $l$ so that $Q^t SPS^t Q = \text{diag}(l)$ and set $\mu := \sigma^2 + l$. We now show $\mu \in \mathbb{L}_C(r,\sigma^2)$. The design $(\bar{S},\bar{P}) = f_\Sigma(S,P)$ satisfies $\bar{S}\bar{P}\bar{S}^t = \Sigma^{-\frac{1}{2}}SPS^t\Sigma^{-\frac{1}{2}}$ which, along with the commutativity of $\Sigma$ and $SPS^t$ implies that $Q^t\bar{S}\bar{P}\bar{S}^tQ = \text{diag}(l/\sigma^2)$ (see (7)). Furthermore, from Lemma 3, $(\bar{S},\bar{P})$ is a tight design for $r$ on $I_N$. Lemma 6 implies that the ordered eigenvalues belong to $\mathbb{L}_I(r)$, i.e.,

$$\left(\left(\frac{l}{\sigma^2}\right)_{[1]}, \left(\frac{l}{\sigma^2}\right)_{[2]}, \cdots, \left(\frac{l}{\sigma^2}\right)_{[N]}\right) \in \mathbb{L}_I(r),$$

and it immediately follows from the definitions of $\mathbb{L}_C\left(r,\sigma^2\right)$ and $\mu$ that $\mu \in \mathbb{L}_C(r,\sigma^2)$.

Conversely, let $\mu \in \mathbb{L}_C(r,\sigma^2)$. Define $\lambda := \mu/\sigma^2 - \mathbf{1}$ where $\mathbf{1} \in \mathbb{R}_+^N$ is the all 1 vector. Observe that $(\lambda_{[1]},\cdots,\lambda_{[N]}) \in \mathbb{L}_I(r)$. Since $\mu = \lambda\cdot\sigma^2 + \sigma^2$, it is sufficient to demonstrate the existence of a tight commuting design $(\tilde{S},\tilde{P})$ for $r$ on $\Sigma$ that satisfies $Q^t\tilde{S}\tilde{P}\tilde{S}^tQ = \text{diag}(\lambda\cdot\sigma^2)$.

Since $\left(\lambda_{[1]},\lambda_{[2]},\cdots,\lambda_{[N]}\right) \in \mathbb{L}_I(r)$, Lemma 6 guarantees the existence of a tight design $\left(\bar{S},\bar{P}\right)$ for $r$ on $I_N$ that satisfies $\text{eig}(\bar{S}\bar{P}\bar{S}^t) = \left(\lambda_{[1]},\lambda_{[2]},\cdots,\lambda_{[N]}\right)$. Pick $U$ so that $U^t\bar{S}\bar{P}\bar{S}^tU = \text{diag}(\lambda)$. As in the proof of Theorem 5, we set $\hat{S} = QU^t\bar{S}$ to yield a tight design $(\hat{S},\bar{P})$ for $r$ on $I_N$ with $Q^t\hat{S}\bar{P}\hat{S}^tQ = \text{diag}(\lambda)$. Consequently, $(\tilde{S},\tilde{P}) = f_{\Sigma^{-1}}(\hat{S},\bar{P})$ is a tight commuting design for $r$ on $\Sigma$. Since $\tilde{S}\tilde{P}\tilde{S}^t = \Sigma^{\frac{1}{2}}\hat{S}\bar{P}\hat{S}^t\Sigma^{\frac{1}{2}}$, and because $\Sigma$ and $\hat{S}\bar{P}\hat{S}^t$ commute, we have $Q^t\tilde{S}\tilde{P}\tilde{S}^tQ = \text{diag}(\lambda\cdot\sigma^2)$, and the proof of the theorem is complete. ■

Because of commutativity, the eigenvectors of $\Sigma$ and the signal correlation matrix match. A further efficiency (in terms of power) is possible if the higher energy directions of the signal correlation matrix align with the less noisy directions. This is summarized below.

*Theorem 8:* Let $r, \Sigma, Q, \sigma^2$ be as in Theorem 7. Define

$$\mathbb{L}(r,\sigma^2) := \left\{ \mu \in \mathbb{R}_+^N : \begin{array}{c} \left(\frac{\mu_i}{\sigma_i^2}\right) \geq 1,\ i \in [\![1, N]\!], \\ \prod_{i=1}^{n}\frac{\mu_i}{\sigma_i^2} \geq \prod_{i=1}^{n}\exp\{2r_{[i]}\} \\ \text{for } n \in [\![1, N-1]\!], \\ \prod_{i=1}^{N}\frac{\mu_i}{\sigma_i^2} = \prod_{i=1}^{K}\exp\{2r_{[i]}\}. \end{array} \right\}. \quad (13)$$

(i) For every $\mu \in \mathbb{L}(r,\sigma^2)$, there exists a tight commuting design $(\tilde{S},\tilde{P})$ for $r$ on $\Sigma$ that satisfies

$$\left(\mu_{[1]},\mu_{[2]},\cdots,\mu_{[N]}\right) = \text{eig}(\Sigma + \tilde{S}\tilde{P}\tilde{S}^t).$$

(ii) If $(\tilde{S},\tilde{P})$ is a design for $r$ on $\Sigma$, then there exists $\mu \in \mathbb{L}(r,\sigma^2)$ that satisfies $\sum_{i=1}^{N}\left(\mu_i - \sigma_i^2\right) \leq \text{tr}(\tilde{P})$. On account of (i), this $\mu$ will lead to a tight commuting design with a lesser or equal sum power.

*Remark 3:* $\mathbb{L}_C(r,\sigma^2)$ completely characterizes $\text{eig}(\Sigma + SPS^t)$ for all tight commuting designs for $r$ on $\Sigma$. However, the smaller set $\mathbb{L}(r,\sigma^2)$ is sufficient for purposes of solving PMIN.

*Proof of Theorem 8:* To prove (i), observe that

$$\prod_{i=1}^{n}\left(\mu/\sigma^2\right)_{[i]} \geq \prod_{i=1}^{n}\left(\mu_i/\sigma_i^2\right), n \in [\![1, N-1]\!].$$

Consequently, $\mathbb{L}(r,\sigma^2) \subseteq \mathbb{L}_C(r,\sigma^2)$, and the existence of a design for $r$ on $\Sigma$ with the desired output covariance matrix follows from Theorem 7.

To prove (ii), we may assume that $(\tilde{S},\tilde{P})$ is a commuting design on account of Theorem 5. Let $Q^t\tilde{S}\tilde{P}\tilde{S}^tQ = \text{diag}(l)$. Then $(S,P) = f_\Sigma(\tilde{S},\tilde{P})$ is a design for $r$ on $I_N$ that satisfies $Q^tSPS^tQ = \text{diag}(l/\sigma^2)$ (from (7)). From Lemma 6,

$$\lambda := \left((l/\sigma^2)_{[1]}, \cdots, (l/\sigma^2)_{[N]}\right) \in \mathbb{L}_I(r),$$

and therefore

$$\mu := (\lambda\cdot\sigma^2 + \sigma^2) \in \mathbb{L}(r,\sigma^2).$$

The vector $\mu$ is the desired vector because

$$\begin{aligned}\text{tr}(\tilde{P}) = \sum_{n=1}^{N} l_n &= \sum_{n=1}^{N}\left(\frac{l_n}{\sigma_n^2}\right)\sigma_n^2 \\ &\geq \sum_{n=1}^{N}\left(\frac{l}{\sigma^2}\right)_{[n]}\sigma_n^2 \qquad (14)\\ &= \sum_{n=1}^{N}\lambda_n\sigma_n^2 = \sum_{n=1}^{N}\left(\mu_n - \sigma_n^2\right)\end{aligned}$$

where (14) follows from (9) applied to the diagonal matrices $\text{diag}(\sigma^2)$ and $\text{diag}(l/\sigma^2)$. ■

### D. *Separable convex optimization and RMAX-PMIN Duality*

We saw that PMIN can be solved by focusing on tight commuting designs (Section II-B). We also obtained a characterization of such designs (Section II-C). We now reduce PMIN to a separable convex optimization problem with linear *ascending* inequality and equality constraints. Subsequently, we briefly discuss the duality between RMAX and PMIN.

*Theorem 9:* PMIN can be stated as

$$\min\left\{\sum_{n=1}^{N}\sigma_n^2\exp\{2x_n\}\right\} - \text{tr}(\Sigma) \qquad (15)$$

subject to

$$\begin{aligned} x_i &\geq 0,\ i\in[\![1,N]\!],\\ \sum_{i=1}^{n}x_i &\geq \sum_{i=1}^{n}\gamma_{[i]},\ n\in[\![1,N-1]\!],\\ \sum_{i=1}^{N}x_i &= \sum_{i=1}^{K}\gamma_{[i]}. \qquad (16)\end{aligned}$$

*Proof:* We already reduced PMIN to

$$\min\left\{\sum_{n=1}^{N}\mu_n : \mu\in\mathbb{L}(r,\sigma^2)\right\} - \text{tr}(\Sigma).$$

The bijective transformation $x = \frac{1}{2}\log(\mu/\sigma^2)$ and $\gamma = r$ transform the constraints in $\mathbb{L}(r,\sigma^2)$ to the linear constraints in (16). ■

Such separable convex optimization problems with linear ascending inequality constraints were solved in [4]. The objective function is separable because $\sum_{n=1}^{N}\sigma_n^2\exp\{2x_n\}$ separates into a sum of $N$ functions, each of one variable. Algorithm 1 of [4] solves the problem.

As a final remark, we observe that with the mapping $\tilde{\sigma}^2 = \frac{1}{2}\log\sigma^2$, [4, Algorithm 1] is identical to [2, Algorithm $\mathcal{A}$] (with $\tilde{\sigma}^2$ input to algorithm instead of $\sigma^2$). The objective functions are different, but the steps to put out the optimum vector are identical. The algorithm [2, Algorithm $\mathcal{A}$] maximises

$$\sum_{n=1}^{N}\frac{1}{2}\log\left(1+\frac{x_n}{\sigma_n^2}\right) \qquad (17)$$

under the positivity, inequality and equality constraints in (16). The objective function in (15) is minimized and (17) maximized simultaneously by the same so-called *Schur-minimal* element [2, p.1299], i.e., an element $x^*$ that satisfies the constraints in (16) and is such that if an $x$ satisfies the constraints in (16), then $\sum_{i=1}^{n}x_{[i]} \geq \sum_{i=1}^{n}x^*_{[i]}$ for each $n\in[\![1,N]\!]$. The bijective transformation given in the proof of Theorem 9 enables us to go from a solution for RMAX to a solution for PMIN, and vice-versa.

## III. Sequence Design Algorithms

### A. *PMIN*

In this section we present an algorithm to output sequences and powers that solve PMIN. One reason for this new algorithm is that it provides a bound on the number of distinct sequences used (Section III-C). We would like to keep the number of distinct sequences small to reduce downlink signaling in a dynamic environment. Indeed, in our centralized framework, the uplink receiver should compute the sequences and powers to be employed by users for uplink transmission, and should signal these to the users on the opposite link (downlink). If the number of distinct sequences is small, and Theorem 16 later shows we need at most $2N-1$ regardless of the number of users $K$, then the uplink receiver may broadcast these $2N-1$ sequences and simply indicate which of these each user should employ. Each such index requires $\lceil\log_2(2N-1)\rceil$ bits. In a dynamic environment, where users may enter and leave the system, the new sequences and powers may need to be re-signaled often. If the number of distinct sequences is small, then our suggested sequence-broadcast method provides savings in signaling when $K$ is large. This motivates the need for as few distinct sequences as possible. Our work may also be useful in some signal processing applications where *tight frames* [18] with fewest vectors are desired.

We now discuss how to generate the sequences. Our first observation is that we may restrict attention to diagonal $\Sigma$, i.e., $\Sigma = \text{diag}(\sigma^2)$. The noise components are independent Gaussians but not necessarily of equal variance. A simple orthonormal transformation can then take a design on $\text{diag}(\sigma^2)$ to a design on an arbitrary $\Sigma$ with the same set of eigenvalues.

Our second observation is that if the $N$-length all-equal vector $\bar{\mu} = (\mu_{\max}, \mu_{\max}, \cdots, \mu_{\max})$ belongs to $\mathbb{L}\left(r, \sigma^2\right)$, where

$$\mu_{\max} := \left( \exp\left\{ 2 \sum_{k=1}^{K} r_k \right\} \prod_{i=1}^{N} \sigma_i^2 \right)^{\frac{1}{N}}, \qquad (18)$$

then it has minimum sum by arithmetic-mean geometric-mean inequality. It is therefore the desired Schur-minimal element in $\mathbb{L}\left(r, \sigma^2\right)$ [2, Section II-B].

In general, the all-equal vector may not lie in $\mathbb{L}\left(r, \sigma^2\right)$. However, our third observation on a feature of [2, Algorithm $\mathcal{A}$] and [4, Algorithm 1] enables us restrict attention to such vectors. Indeed, let $\mu^* \in \mathbb{L}\left(r, \sigma^2\right)$ solve PMIN. Then the above algorithms partition the user space $\{1, 2, \cdots, K\}$, and the available dimensions $\{1, 2, \cdots, N\}$ with noise variances $\sigma_1^2, \sigma_2^2, \cdots, \sigma_N^2$, into $M$ user subsets $J_1, J_2, \cdots, J_M$ and $M$ dimension subsets $D_1, D_2, \cdots, D_M$ respectively. We refer to a user subset $J_m$ and it's associated dimension subset $D_m$ as a *subsystem*. Users in group $J_m$ are associated with the subspace span$\{e_i : i \in D_m\}$ and will signal only within this subspace. The additional property of these partitions is that $(\mu^*)_i = \bar{\nu}_m$ for all $i \in D_m$. This desired feature enables us to interpret $\bar{\nu}_m$ as a common water-level for users in $J_m$. We then restrict attention to users in a subsystem, and assign sequences and powers subsystem-by-subsystem. The assignment problem for a group $m$ is a similar but smaller problem with $K$ replaced by $|J_m|$ and $N$ replaced by $|D_m|$, with the desired feature that the constant vector $\bar{\nu}^{(m)} := (\nu_m : i \in D_m)$ is the Schur-minimal element of $\mathbb{L}\left((r_k : k \in J_m), (\sigma_i^2 : i \in D_m)\right)$.

We thus arrive at the useful conclusion that we may restrict attention to sequence and power allocations to those (sub)systems with $K$ users, $N$ dimensions, and $\bar{\mu} \in \mathbb{L}\left(r, \sigma^2\right)$, where $\bar{\mu}$ is given by (18).

We next assign sequence and power to users, one after another[2]. Assume $k-1$ users are already added. To add the $k$th user, a desired set of eigenvalues $\lambda^{(k)} \in \mathbb{R}^N$ of the $k$-user signal and noise matrix $A^{(k)} := \Sigma + \sum_{i=1}^{k} p_k s_k s_k^t$ is first found. (We define $A^{(0)} := \Sigma$). To meet the rate requirement $r_k$, the set of eigenvalues $\lambda^{(k)} = \text{eig}\left(A^{(k)}\right) = \text{eig}\left(A^{(k-1)} + p_k s_k s_k^t\right)$ should satisfy

$$r_k = \frac{1}{2} \log \frac{\left|A^{(k-1)} + p_k s_k s_k^t\right|}{\left|A^{(k-1)}\right|} = \frac{1}{2} \log \frac{\prod_{n=1}^{N} \lambda_n^{(k)}}{\prod_{n=1}^{N} \lambda_n^{(k-1)}}. \qquad (19)$$

Such an assignment will put us on the vertex of a particular capacity region. (See proof of Theorem 13 later). To solve for $p_k$ and $s_k$, we could assign $\lambda^{(k)}$ such that $\lambda^{(k)}$ and $\lambda^{(k-1)}$ satisfy an *interlacing* property, as given in the following sufficiency lemma.

*Lemma 10:* [16, Proposition 1] Let $A$ be an $N \times N$ real symmetric matrix with eig$(A) = \lambda$. Let $\hat{\lambda} \in \mathbb{R}^N$ and $\lambda$ interlace, i.e.,

$$\hat{\lambda}_1 \geq \lambda_1 \geq \hat{\lambda}_2 \geq \lambda_2 \geq \cdots \geq \hat{\lambda}_N \geq \lambda_N. \qquad (20)$$

Then, there exists a $v = f\left(A, \hat{\lambda}\right)$, an element of $\mathbb{R}^N$, such that eig$\left(A + vv^t\right) = \hat{\lambda}$.

[2]Within the subsystem, the ordering of the users does not matter.

See [16, Appendix I] for a proof. A preliminary version of this result appeared in [19]. Chu & Chu provide a similar result for singular values [20, Thm 2.3].

We now provide an algorithm that satisfies (19) and (20) for every user index.

*Algorithm 11:* **Identifies $(S, P)$ that solves PMIN:**

$(S,P) = \mathit{SolvePMIN}\left(r, \sigma^2, K, N\right)$ {
  $\left(\lambda^{(k)} : k = 1, \cdots, K\right) \leftarrow \mathit{InterlaceEigenValues}(r, \sigma^2, K, N)$
  $(S, P) \leftarrow \mathit{AssignSequencesPowers}(\lambda^{(k)} : k = 1, \cdots, K)$
}

**Subroutine to identify intermediate eigenvalues :**

$\left(\lambda^{(k)} : k = 1, \cdots, K\right) = \mathit{InterlaceEigenValues}(r, \sigma^2, K, N)$ {
  $\mu_{\max} \leftarrow \left(\exp\left\{2 \sum_{k=1}^{K} r_k\right\} \prod_{i=1}^{N} \sigma_i^2\right)^{\frac{1}{N}}$
  $d \leftarrow 1$
  *for* $(n \in [\![1, N]\!])$
    $\lambda_n^{(0)} \leftarrow \sigma_{N-n+1}^2$
  *for* $(k = 1, \cdots, K)$ {
    $n^*(k) \leftarrow \min \mathcal{F}(k) := \left\{ n \in [\![d, N]\!] : \lambda_n^{(k-1)} \exp\{2r_k\} \leq \mu_{\max} \right\}$ *(a)*
    *if* $(n^*(k) = d)$ {
      $\lambda_{n^*(k)}^{(k)} \leftarrow \lambda_{n^*(k)}^{(k-1)} \cdot \exp\{2r_k\}$ *(b)*
      *for* $(n \in [\![1, N]\!] \setminus \{n^*(k)\})$
        $\lambda_n^{(k)} \leftarrow \lambda_n^{(k-1)}$ *(c)*
      *if* $\left(\lambda_{n^*(k)}^{(k-1)} \cdot \exp\{2r_k\} = \mu_{\max}\right)$
        $d \leftarrow d + 1$ *(d)*
    }
    *else* {
      $\lambda_{n^*(k)-1}^{(k)} \leftarrow \mu_{\max}$ *(e)*
      $\lambda_{n^*(k)}^{(k)} \leftarrow \lambda_{n^*(k)-1}^{(k-1)} \cdot \frac{\lambda_{n^*(k)}^{(k-1)} \cdot \exp\{2r_k\}}{\mu_{\max}}$ *(f)*
      *for* $(n \in [\![1, N]\!] \setminus \{n^*(k) - 1, n^*(k)\})$
        $\lambda_n^{(k)} \leftarrow \lambda_n^{(k-1)}$ *(g)*
      $\lambda^{(k)} \leftarrow \mathit{SortDescending}\left(\lambda^{(k)}\right)$ *(h)*
      $d \leftarrow d + 1$ *(i)*
    }
  }
  *RETURN* $\left(\lambda^{(k)} : k = 1, \cdots, K\right)$
}

**Subroutine to identify sequences and powers :**

$(S, P) = \mathit{AssignSequencesPowers}\left(\lambda^{(k)} : k = 1, \cdots, K\right)$ {
  $A^{(0)} \leftarrow \Sigma$
  *for* $(k = 1, 2, \cdots, K)$ {
    $c_k \leftarrow f\left(A^{(k-1)}, \lambda^{(k)}\right)$
    $s_k \leftarrow c_k / \|c_k\|$
    $p_k \leftarrow \|c_k\|$
    $A^{(k)} \leftarrow A^{(k-1)} + p_k s_k s_k^t$
  }
  *RETURN* $(S, P)$
}

□

Subroutine *SortDescending*$(c)$ returns a vector with components in descending order.

We now provide a discussion on subroutine *InterlaceEigenValues*. The set of eigenvalues $\lambda^{(0)}$ is initialized to the noise variances in decreasing order. We then compute a vector sequence of eigenvalues using this subroutine, one after another. To compute $\lambda^{(k)}$, we identify the largest component of $\lambda^{(k-1)}$ that can accommodate user $k$'s rate requirement $r_k$ without exceeding $\mu_{\max}$. Since $\lambda^{(k-1)}$ is in descending order, it is the smallest index $n$ such that $\lambda_n^{(k-1)} \exp\{2r_k\} \leq \mu_{\max}$ and is denoted $n^*(k)$ (see (a)). The parameter $d$ indexes the largest component of $\lambda^{(k-1)}$ that is lesser than $\mu_{\max}$, and so the search for $n^*(k)$ maybe restricted to indices in $[\![d, N]\!]$.

If it so happens that index $d$ itself can accommodate user $k$, i.e., $n^*(k)$ equals $d$, just this component of $\lambda^{(k)}$ is raised as per (19) to meet user $k$'s requirement $r_k$ (see (b)). All other components of $\lambda^{(k)}$ remain the same as that of $\lambda^{(k-1)}$ (see (c)). It is possible that the updated component has already reached its maximum value $\mu_{\max}$ in which case $d$ is incremented (see (d)).

If the $d^{\text{th}}$ component cannot accommodate user $k$'s rate $r_k$, we raise component $n^*(k) - 1$ to $\mu_{\max}$ (see (e)) and component $n^*(k)$ to an appropriate value (see (f)) so that user $k$'s rate is indeed accommodated (see (19)). At this stage, only two eigenvalues are updated. The subsequent *SortDescending* takes the component at $n^*(k) - 1$ to location $d$, and components in $d, d+1, \cdots, n^*(k) - 2$ one step to the right. Exactly one component is raised to its maximum value $\mu_{\max}$ and so $d$ is incremented (see (i)).

The vector obtained after application of *SortDescending* has an interesting water-filling interpretation. $\frac{1}{2}\log\bar{\mu}$ could be thought of as a common "water-level" in all dimensions. Each user comes with a volume $r_k$ which gets poured in columns with existing heights $\frac{1}{2}\log\lambda^{(k-1)}$ to get new heights $\frac{1}{2}\log\lambda^{(k)}$ as follows : pour it to raise the tallest column not already at level $\frac{1}{2}\log\mu_{\max}$ to this maximum level; then raise others to levels equaling the respectively previous column's initial height, until all $r_k$ is poured. Such a filling ensures interlacing property of $\lambda^{(k)}$ and $\lambda^{(k-1)}$ is preserved, and rate $r_k$ for user $k$ can be supported.

While *AssignSequencesPowers*$(\cdot)$ makes repeated use of Lemma 10, some further remarks on the correctness of the subroutine *InterlaceEigenValues*$(\cdot)$ are needed. These are summarized in the following.

*Lemma 12:* If $\bar{\mu}$ defined via (18) belongs to $\mathbb{L}\left(r, \sigma^2\right)$, then the following hold for the subroutine *InterlaceEigenValues*$(\cdot)$ and its output $\left(\lambda^{(k)} : k \in [\![1, K]\!]\right)$ :

(1) In every iteration, the set

$$\mathcal{F}(k) := \left\{ n \in [\![1, N]\!] : \lambda_n^{(k-1)} \exp\{2r_k\} \leq \mu_{\max} \right\}$$

is nonempty;

(2) $\lambda^{(k)}$ and $\lambda^{(k-1)}$ interlace;

(3) $\lambda^{(k)}$ and $\lambda^{(k-1)}$ satisfy (19); and

(4) $\lambda^{(K)} = \bar{\mu}$.

*Proof:* See Appendix A. ■

The following theorem guarantees the correctness of Algorithm 11.

*Theorem 13:* If $\bar{\mu}$ defined via (18) belongs to $\mathbb{L}\left(r, \sigma^2\right)$, then *SolvePMIN* puts out the sequences and powers that solve PMIN.

*Proof:* It is clear that $\bar{\mu}$ is the desired vector in $\mathbb{L}\left(r, \sigma^2\right)$. Lemma 12 shows that the subroutine *InterlaceEigenValues* runs to completion and puts out a vector sequence $\left(\lambda^{(k)} : k \in [\![1, K]\!]\right)$ whose consecutive vectors interlace and $\lambda^{(K)} = \bar{\mu}$. *AssignSequencesPowers*, thanks to Lemma 10, then puts out a design $(S, P)$. Finally $r = (r_1, r_2, \cdots, r_K)$ satisfies (19) and is a vertex of a polymatroidal polyhedron $C(S, P, \Sigma)$. Such polyhedra contain all their vertices, i.e., $r \in C(S, P, \Sigma)$, and $(S, P)$ is a design for $r$ on $\Sigma$. ■

*Remark 4:* The more general case when $\bar{\mu} \notin \mathbb{L}\left(r, \sigma^2\right)$ is solved by reducing the problem to smaller ones that satisfy the hypothesis of Theorem 13, as outlined in the beginning of the section.

### B. RMAX

The dual problem considered by Viswanath & Anantharam [2] is :

**RMAX** : Given $p \in \mathbb{R}_+^K$ and a pd $\Sigma$ of size $N \times N$, let $P =$ diag$(p)$. Find

$$R_{\max} = \max_{S \in \mathcal{S}} \left\{ \sum_{k=1}^{K} r_k : r \in C(S, P, \Sigma) \right\}.$$

□

Viswanath & Anantharam [2] show that

$$R_{\max} = \max \left\{ \frac{1}{2} \sum_{n=1}^{N} \log \frac{\mu_n}{\sigma_n^2} : \mu \in \mathcal{L}(p, \sigma^2) \right\}.$$

where

$$\mathcal{L}(p, \sigma^2) = \left\{ \mu \in \mathbb{R}_+^N : \begin{array}{c} \mu_i - \sigma_i^2 \geq 0, \ i \in [\![1, N]\!] \\ \sum_{i=1}^{n} \mu_i - \sigma_i^2 \geq \sum_{i=1}^{n} p_{[i]}, \\ \text{for } n \in [\![1, N-1]\!], \\ \sum_{i=1}^{N} \mu_i - \sigma_i^2 = \sum_{i=1}^{K} p_{[i]}. \end{array} \right\}. \tag{21}$$

The following algorithm solves RMAX when $\bar{\mu} \in \mathcal{L}(p, \sigma^2)$, where $\bar{\mu} = (\mu_{\max}, \mu_{\max}, \cdots, \mu_{\max})$, and

$$\mu_{\max} = \frac{\sum_{k=1}^{K} p_k + \sum_{n=1}^{N} \sigma_n^2}{N}. \tag{22}$$

This is a consequence of the tight duality that reduces one problem to the other. The proof is easy and follows the arguments of the previous section. Indeed, the constraints in (13) and (21) are identical and both search for Schur-minimal elements in the set, as indicated in the previous section.

*Algorithm 14:* **Identifies** $(S, r)$ **that solves RMAX :**

$(S, P) =$ *SolveRMAX*$\left(p, \sigma^2, K, N\right)$ {
$\quad\left(\exp\left\{2\lambda^{(k)}\right\} : k = 1, \cdots, K\right)$
$\qquad\leftarrow$ *InterlaceEigenValues*$\left(p, \exp\left\{2\sigma^2\right\}, K, N\right)$
$\quad(S, r) \leftarrow$ *AssignSequencesRates*$\left(\left(\lambda^{(k)} : k = 1, \cdots, K\right), p\right)$
}

**Subroutine to identify sequences and rates :**

$(S, r) = \mathit{AssignSequencesRates}\big(\lambda^{(k)} : k = 1, \cdots, K, p\big)$ {
  $A^{(0)} \leftarrow \Sigma$
  *for* $(k = 1, 2, \cdots, K)$ {
    $c_k \leftarrow f\left(A^{(k-1)}, \lambda^{(k)}\right)$
    $s_k \leftarrow c_k / \|c_k\|$
    $A^{(k)} \leftarrow A^{(k-1)} + p_k s_k s_k^t$
    $r_k \leftarrow \frac{1}{2}\log\left|A^{(k)}\right| - \frac{1}{2}\log\left|A^{(k-1)}\right|$
  }
  *RETURN* $(S, r)$
}

□

### C. Bounds on number of sequences and sufficiency of $N$ orthogonal sequences

We study the size of optimal sequence sets output by *SolvePMIN* and *SolveRMAX*. We begin by a simple characterization of rank-1 perturbations that result in the change of a single eigenvalue.

*Proposition 15:* Let A be an $N \times N$ real symmetric matrix with $\text{eig}(A) = \lambda = (\lambda_1, \cdots, \lambda_N)$. Let $U$ be an orthonormal matrix that diagonalizes $A$, *i.e.*, $A = U\Lambda U^t$, where $\Lambda = \text{diag}\{\lambda\}$. Fix $l$ satisfying $1 \le l \le N$. Let $\hat{\lambda}$ be $N$ real numbers such that $\hat{\lambda}_j = \lambda_j$ for all $j$ except when $j = l$, and $\hat{\lambda}_l \ge \lambda_l$. Furthermore, let $\hat{\lambda}$ and $\lambda$ satisfy the interlacing property (20). Then

(i) with $v = \left(\sqrt{\hat{\lambda}_l - \lambda_l}\right) u_l$, where $u_l$ is the $l$th column of $U$, we have $\text{eig}(A + vv^t) = \hat{\lambda}$;

(ii) for the choice of $v$ in (i), $A$ and $(A + vv^t)$ are simultaneously diagonalized by $U$.

*Proof:* Let $U = [u_1 \ \cdots \ u_N]$, where $u_i \in \mathbb{R}^N$. Then $Au_l = \lambda_l u_l$, and $u_l^t u_i = 0$ if $i \ne l$. Hence for every $i \ne l$,

$$\left(A + vv^t\right) u_i = Au_i + v\left(v^t u_i\right) = Au_i + 0 = \lambda_i u_i,$$

and thus $\lambda_i, i \ne l$, are $N - 1$ eigenvalues of $A + vv^t$. Furthermore,

$$\left(A + vv^t\right) u_l = \lambda_l u_l + \left(\hat{\lambda}_l - \lambda_l\right) u_l(u_l^t u_l) = \hat{\lambda}_l u_l,$$

and thus $\hat{\lambda}_l$ is an eigenvalue of $A + vv^t$. This proves the first statement.

The set of eigenvectors for the two real symmetric matrices are identical and therefore the orthonormal matrix composed of these eigenvectors simultaneously diagonalizes them. ■

As a consequence of Proposition 15, we infer the following.

*Theorem 16: SolvePMIN* and *SolveRMAX* put out sequence sets with at most $2N - 1$ distinct sequences.

*Proof:* Let users be assigned sequences in increasing order of user indices. Let $j_1 < j_2 < \cdots < j_N$ be user indices such that rate requirement of user $j_n$ necessitates one component of $\lambda^{(j_n - 1)}$ to be raised to $\mu_{\max}$. In other words, $d$ in *InterlaceEigenValues* is incremented in the iteration corresponding to users $j_n : n \in [\![1, N]\!]$. The addition of users in $[\![j_{n-1} + 1, j_n - 1]\!]$ for $n \in [\![1, N-1]\!]$ ($j_0 := 1$) result in raising just one eigenvalue, and indeed the one with index $d$. This is the component with the largest eigenvalue strictly below $\mu_{\max}$. From Proposition 15 all these users are assigned the same sequence. This gives at most $N - 1$ distinct sequences for all such users. Each user $j_n : n \in [\![1, N-1]\!]$ is assigned a possibly distinct sequence, thus contributing at most another $N - 1$ distinct sequences. Finally, users in $[\![j_{N-1} + 1, N]\!]$ raise just one eigenvalue (there is just one eigenvalue now that is less than $\mu_{\max}$). From Proposition 15, they contribute at most another distinct sequence. Summing over the number of possible distinct sequences, the upper bound is proved. ■

The proof of Theorem 16 suggests a technique to restrict the number of distinct sequences to $N$ if multi-dimensional signaling is allowed. Suppose each user $j_n : n \in [\![1, N]\!]$ raises exactly one eigenvalue to $\mu_{\max}$. This happens when $n^*(j_n) = d$ and $\lambda^{(j_n - 1)}_{n^*(j_n)} \exp\{2r_{j_n}\} = \mu_{\max}$. Then each user in $[\![j_{n-1} + 1, j_n]\!]$ for $n \in [\![1, N]\!]$ ($j_0 := 1$ as before) raises just one eigenvalue; indeed, the eigenvalue with index $d$. From Proposition 15 these users are assigned the same sequence resulting in just $N$ distinct sequences. Furthermore, it can be inferred from Proposition 15(ii), that the assigned sequences are columns of the orthonormal matrix that diagonalizes $A^{(0)}$.

In general, each user $j_n : n \in [\![1, N]\!]$ raises two eigenvalues prior to the execution of the *SortDescending* subroutine. A simple rate-splitting technique enables us to cast the rate requirement vector $r$ as $r'$ where the above property holds. It is easy to see that at most $N$ components of $r$ are split. Users corresponding to components that are split are assigned multiple sequences. The following theorem bounds the number of components into which a component is split. We state the theorem for PMIN. An analogous result holds for RMAX. We refer the reader to [21] for a proof.

*Theorem 17:* Let the all-constant vector $\bar{\mu}$ given via (18) belong to $\mathbb{L}\left(r, \sigma^2\right)$. The rate requirement vector $r \in \mathbb{R}_+^K$ can be cast into a rate requirement vector $r' \in \mathbb{R}_+^{K'}$, $K \le K' \le K + N - 1$, by splitting at most $N - 1$ components into two components each such that the output sequence assignment of *SolveRMAX* to $r'$ forms an orthonormal set.

Observe that when the all-constant vector does not belong to $\mathbb{L}\left(r, \sigma^2\right)$, then (as observed in Section III-A), the problem can be broken down into $M$ subproblems each with processing gain $|D_m|$ such that the all-constant vector belongs to the constraint set for each of these subproblems. Repeated application of Theorem 17 to each of these subproblems implies that the rate requirement $r \in \mathbb{R}_+^K$ can be cast into $r' \in \mathbb{R}_+^{K'}$, $K \le K' \le K + N - M$, by splitting at most $N - M$ components into two components each such that the output sequence assignment of *SolveRMAX* to $r'$ forms an orthonormal set.

As a final remark, employing a rate splitting technique based on Theorem 17 results in the optimality of any orthonormal sequence set. In particular, we may employ the standard basis vectors. This restricts the sequences to take values in a finite alphabet.

## APPENDIX A
## PROOF OF LEMMA 12

We begin by stating the hypothesis of the lemma. From the decreasing order of $\lambda^{(0)}$, we have

$$\prod_{i=1}^{n} \frac{\mu_{\max}}{\lambda^{(0)}_{N-i+1}} \ge \prod_{i=1}^{n} \exp\left\{2r_{[i]}\right\} \tag{23}$$

for each $n \in [\![1, N-1]\!]$, and

$$\prod_{i=1}^{N} \frac{\mu_{\max}}{\lambda_{N-i+1}^{(0)}} = \prod_{i=1}^{K} \exp\left\{2r_{[i]}\right\}. \quad (24)$$

(1) Our proof is by induction on $N$.

Case $N = 1$ : Since $\bar{\mu} \in \mathbb{L}\left(r, \sigma^2\right)$, we have $\mu_{\max} = \lambda_1^{(0)} \prod_{k=1}^{K} \exp\left\{2r_k\right\}$. As a consequence,

$$\underbrace{\lambda_1^{(0)} \exp\{2\sum_{k=1}^{j-1} r_k\}}_{\lambda_1^{(j-1)}} \exp\left\{2r_j\right\} \leq \mu_{\max}$$

holds for all $j \in [\![1, K]\!]$, i.e., $\mathcal{F}(k) = \{1\}$ for all $k \in [\![1, K]\!]$.

From (23) with $n = 1$, we have $N \in \mathcal{F}(1)$. Let $j$ be the first user who when added makes $1 \notin \mathcal{F}(j)$, i.e., $\lambda_1^{(k-1)} \exp\left\{2r_k\right\} \leq \mu_{\max}$ for $k \in [\![1, j-1]\!]$, and $\lambda_1^{(j-1)} \exp\left\{2r_j\right\} > \mu_{\max}$. Then for the first $j-1$ users, $\lambda_N^{(k)} = \sigma_1^2$, $k \in [\![1, j-1]\!]$. Applying (23) again with $n = 1$, we have $N \in \mathcal{F}(k), k \in [\![1, j]\!]$. So $\mathcal{F}(j)$ is nonempty, and $n^*(j)$ is well defined.

User $j$ is the first user for whom a *SortDescending* call may return an output different from the input. Indeed, let $l = n^*(j)$. Then *SortDescending*$\left(\lambda^{(j)}\right)$ returns

$$\begin{aligned}\lambda^{(j)} &= \Big(\mu_{\max}, \beta, \lambda_2^{(j-1)}, \lambda_3^{(j-1)}, \cdots \\ &\qquad \cdots, \lambda_{l-2}^{(j-1)}, \alpha, \lambda_{l+1}^{(j-1)}, \cdots, \lambda_N^{(j-1)}\Big) \quad (25)\\ &= \left(\mu_{\max}, \beta, \lambda_2^{(0)}, \lambda_3^{(0)}, \cdots, \lambda_{l-2}^{(0)}, \alpha, \lambda_{l+1}^{(0)}, \cdots, \lambda_N^{(0)}\right),\end{aligned}$$

where

$$\beta = \lambda_1^{(0)} \exp\left\{2\sum_{k=1}^{j-1} r_k\right\} = \sigma_N^2 \exp\left\{2\sum_{k=1}^{j-1} r_k\right\} \quad (26)$$

and

$$\alpha = \frac{\lambda_{l-1}^{(j-1)} \lambda_l^{(j-1)} \exp\left\{2r_j\right\}}{\mu_{\max}}. \quad (27)$$

The ordering is as indicated because $\lambda_l^{(j)}$ is set to $\mu_{\max}$ and is returned as the maximum. Moreover,

$$\lambda_{l-2}^{(j-1)} \geq \frac{\lambda_{l-1}^{(j-1)} \lambda_l^{(j-1)} \exp\left\{2r_j\right\}}{\mu_{\max}} = \alpha \geq \lambda_{l+1}^{(j-1)}$$

on account of $l = n^*(j)$, being the least index in the nonempty $\mathcal{F}(j)$ and the descending order of the components of $\lambda^{(j-1)}$. This justifies the position of $\alpha$. We next prove $\bar{\mu}' := (\mu_{\max}, \mu_{\max}, \cdots, \mu_{\max}) \in \mathbb{R}_+^{N-1}$ lies in $\mathbb{L}\left(v, \eta^2\right)$, where $v = (r_k : k \in [\![j+1, K]\!])$ and

$$\eta^2 = \left(\beta, \lambda_2^{(j-1)}, \lambda_3^{(j-1)}, \cdots, \lambda_{l-2}^{(j-1)}, \alpha, \lambda_{l+1}^{(j-1)}, \cdots, \lambda_N^{(j-1)}\right)$$

is the $N-1$-length vector with components of $\lambda^{(j)}$ (25) less than $\mu_{\max}$. This will prove the inductive step and validate statement (1).

The components of $\eta^2$ are in decreasing order. To prove $\bar{\mu}' \in \mathbb{L}\left(v, \eta^2\right)$, we need to show

$$\prod_{i=1}^{n} \frac{\mu_{\max}}{\eta_{N-i}^2} \geq \prod_{i=1}^{n} \exp\left\{2v_{[i]}\right\} \quad (28)$$

holds for $n \in [\![1, N-2]\!]$ and

$$\prod_{i=1}^{N-1} \frac{\mu_{\max}}{\eta_{N-i}^2} = \prod_{i=1}^{K-j} \exp\left\{2v_{[i]}\right\}. \quad (29)$$

Equations (25), (26), and (27) imply (29). For $n \in [\![1, N-l]\!]$, $\eta_{N-n}^2 = \lambda_{N-n+1}^{(0)}$ and thus

$$\begin{aligned}\prod_{i=1}^{n} \frac{\mu_{\max}}{\eta_{N-i}^2} = \prod_{i=1}^{n} \frac{\mu_{\max}}{\lambda_{N-i+1}^{(0)}} &\geq \prod_{i=1}^{n} \exp\left\{2r_{[i]}\right\} \\ &\geq \prod_{i=1}^{n} \exp\left\{2v_{[i]}\right\}.\end{aligned}$$

For $n \in [\![N-l+1, N-2]\!]$, (28) is given by

$$\begin{aligned}\prod_{i=1}^{n} \frac{\mu_{\max}}{\eta_{N-i}^2} &= \prod_{i=1}^{N-l} \frac{\mu_{\max}}{\lambda_{N-i+1}^{(0)}} \cdot \frac{\mu_{\max}}{\alpha} \cdot \prod_{i=N-l+3}^{n+1} \frac{\mu_{\max}}{\lambda_{N-i+1}^{(0)}} \\ &\overset{(a)}{=} \frac{1}{\exp\left\{2r_j\right\}} \prod_{i=1}^{n+1} \frac{\mu_{\max}}{\lambda_{N-i+1}^{(0)}} \\ &\overset{(b)}{\geq} \frac{1}{\exp\left\{2r_j\right\}} \prod_{i=1}^{n+1} \exp\left\{2r_{[i]}\right\} \\ &\overset{(c)}{\geq} \prod_{i=1}^{n} \exp\left\{2v_{[i]}\right\}\end{aligned}$$

where (a) follows from substitution of (27), (b) from (23), and (c) because $r_j$ is not a component of $v$. This proves $\bar{\mu}' \in \mathbb{L}\left(\eta^2, v\right)$, statement (1) of Lemma holds from induction.

(2) If $n^*(k) = d$, then $\lambda^{(k)}$ and $\lambda^{(k-1)}$ clearly interlace. It remains to verify the interlacing inequality when $n^*(k) > d$. In this case one of the eigenvalues is set to $\mu_{\max}$, the largest, and therefore it is sufficient to verify

$$\lambda_{n^*(k)-1}^{(k-1)} \geq \lambda_{n^*(k)}^{(k)} \geq \lambda_{n^*(k)}^{(k-1)}.$$

Both of these follow from the assignment for $\lambda_{n^*(k)}^{(k)}$ and the facts $\lambda_{n^*(k)}^{(k-1)} \exp\left\{2r_k\right\} \leq \mu_{\max}$ and $\lambda_{n^*(k)-1}^{(k-1)} \exp\left\{2r_k\right\} > \mu_{\max}$.

(3) Follows immediately from the assignment.

(4) Clearly, $\lambda_n^{(0)} \leq \mu_{\max}$ for each $n \in [\![1, N]\!]$. If $\lambda_n^{(k-1)} \leq \mu_{\max}$ for each $n \in [\![1, N]\!]$, by the assignment in the algorithm and statement (2), $\lambda_n^{(k)} \leq \mu_{\max}$ for each $n \in [\![1, N]\!]$. A repeated use of statement (3) for $k = 1, 2, \cdots, K$, yields

$$\prod_{n=1}^{N} \frac{\lambda_n^{(K)}}{\lambda_n^{(0)}} = \prod_{k=1}^{K} \exp\left\{2r_k\right\}.$$

Using this, (24), and (18), we get $\prod_{n=1}^{N} \lambda_n^{(K)} \geq \mu_{\max}^N$ whence $\lambda^{(K)} = \bar{\mu}$.

## REFERENCES


[1] T. Guess, "CDMA with power control and sequence design: the capacity region with and without multidimensional signaling," *IEEE Trans. Inform. Theory*, vol. 50, no. 11, pp. 2604-2619, Nov. 2004.

[2] P. Viswanath and V. Anantharam, "Optimal sequences for CDMA with colored noise: a Schur-saddle function property," *IEEE Trans. Inform. Theory*, vol. IT-48, pp. 1295-1318, June 2002.

[3] ——, "Optimal sequences and sum capacity of synchronous CDMA systems," *IEEE Trans. Inform. Theory*, vol. 45, no. 6, pp. 1984-1991, Sept. 1999.

[4] A. Padakandla and R. Sundaresan, “Separable convex optimization problems with linear ascending constraints," submitted to *SIAM J. Opt.* [Online], Available: http://arxiv.org/abs/0707.2265, July 2007.

[5] C. Rose, S. Ulukus, and R. Yates, “Wireless systems and interference avoidance," *IEEE Trans. Wireless Commun.*, vol. 1, no. 3, pp. 415-428, July 2002.

[6] J. A. Tropp, I. S. Dhillon, and R. W. Heath Jr., “Finite-step algorithms for constructing optimal CDMA signature sequences," *IEEE Trans. Inform. Theory*, vol. 50, no. 11, pp. 2916-2921, Nov. 2004.

[7] J. Luo, S. Ulukus, and A. Ephremides, “Optimal sequences and sum capacity of symbol asynchronous CDMA systems," *IEEE Trans. Inform. Theory*, vol. 51, no. 8, pp. 2760-2769, Aug. 2005.

[8] J. Kasturi and R. Sundaresan, “Sequence design for symbol-asynchronous CDMA with power or rate constraints," in *Proc. 2007 IEEE COMSWARE - WISARD Workshop*, 2007.

[9] P. Anigstein and V. Anantharam, “Ensuring convergence of the MMSE iteration for interference avoidance to the global optimum," *IEEE Trans. Inform. Theory*, vol. 49, no. 4, pp. 873-885, Apr. 2003.

[10] G. Karystinos and D. Pados, “New bounds on the total squared correlation and optimum design of DS-CDMA binary signature sets," *IEEE Trans. Commun.*, vol. 51, pp. 48-51, Jan. 2003.

[11] ——, “The maximum squared correlation, sum capacity, and total asymptotic efficiency of minimum total-squared-correlation binary signature sets," *IEEE Trans. Inform. Theory*, vol. 51, pp. 348-355, Jan. 2005.

[12] V. Ipatov, “On the Karystinos-Pados bounds and optimal binary DS-CDMA signature ensembles," *IEEE Commun. Letters*, vol. 8, pp. 81-83, Feb. 2004.

[13] C. Ding, M. Golin, and T. Klove, “Meeting the Welch and Karystinos-Pados bounds on DS-CDMA binary signature sets," *Designs, Coding, Cryptography*, vol. 30, Aug. 2003.

[14] F. Vanhaverbeke and M. Moeneclaey, “Binary signature sets for increased user capacity on the downlink of CDMA systems," *IEEE Trans. Wireless Commun.*, vol. 5, no. 7, pp. 1795-1804, July 2006.

[15] B. Rimoldi and R. Urbanke, “A rate-splitting approach to the Gaussian multiple access channel," *IEEE Trans. Inform. Theory*, vol. IT-42, pp. 364-375, Mar. 1996.

[16] R. Sundaresan and A. Padakandla, “The size of optimal sequence sets for synchronous CDMA systems," technical report. [Online]. Available: http://arxiv.org/abs/cs.IT/0606015, June 2006.

[17] A. W. Marshall and I. Olkin, *Inequalities: Theory of Majorization and its Applications*. New York: Academic, 1979.

[18] P. Casazza and J. Kovačević, “Equal-norm tight frames with erasures," *Advances Computational Mathematics*, vol. 18, no. 2-4, pp. 387-430, Feb. 2003.

[19] R. Sundaresan, “On sequence design for a synchronous CDMA channel," in *Proc. 2004 IEEE Int. Symp. Inform. Theory*, 2004.

[20] D. Chu and M. Chu, “Low rank update of singular values," *Mathematics Computation*, vol. 75, no. 255, pp. 1351-1366, July 2006.

[21] A. Padakandla, “Interference management for vector Gaussian multiple access channels," master’s thesis, Indian Institute of Science, Bangalore, India, June 2008.

**Arun Padakandla** received his MSc(Engg) degree from the Dept. of Electrical Communication Engineering, Indian Institute of Science in 2008. He is currently pursuing a PhD at the Dept. of Electrical Engineering and Computer Science, Univ. of Michigan, Ann Arbor. His interests are in multiuser information theory, space time codes, and signal processing.

**Rajesh Sundaresan** (S’96-M’2000-SM’2006) received his B.Tech. degree in electronics and communication from the Indian Institute of Technology, Madras, the M.A. and Ph.D. degrees in electrical engineering from Princeton University, NJ, in 1996 and 1999, respectively. From 1999 to 2005, he worked at Qualcomm Inc., Campbell, CA, on the design of communication algorithms for WCDMA and HSDPA modems. Since 2005 he has been an Assistant Professor in the Electrical Communication Engineering department at the Indian Institute of Science, Bangalore. His interests are in the areas of wireless communication, multiple access systems, information theory, and signal processing for communication systems.