Lecture 28: Random walks

1 GI/GI/1 Queueing Model

Consider a GI/GI/1 queue. Customers arrive in accordance with a renewal process having an arbitrary interarrival distribution \( F \), and the service distribution \( G \).

**Proposition 1.1.** Let \( D_n \) be the delay in the queue of the \( n \)th customer in a GI/GI/1 queue with independent inter-arrival times \( X_n \) and service times \( Y_n \). Let \( S_n \) be a random walk with iid steps \( U_n = Y_n - X_{n+1} \) for all \( n \in \mathbb{N} \). Then, we can write

\[
\Pr\{D_n + 1 \geq c\} = \Pr\{S_j \geq c, \text{ for some } j \in [n]\}. \tag{1}
\]

**Proof.** The following recursion for \( D_n \) is easy to verify

\[
D_{n+1} = (D_n + Y_n - X_{n+1})1_{\{D_n + Y_n - X_{n+1} \geq 0\}} = \max\{0, D_n + U_n\}
\]

Iterating the above relation with \( D_1 = 0 \) yields

\[
D_{n+1} = \max\{0, U_n + \max\{0, D_{n-1} + U_{n-1}\}\} = \max\{0, U_n, U_n + U_{n-1} + D_{n-1}\}
\]

For the random walk \( S_n \) with steps \( U_n \), we can write delay in terms of random walk \( S_n \) as

\[
D_{n+1} = \max\{0, S_n - S_{n-1}, S_n - S_{n-2}, \ldots, S_n - S_0\}
\]

Using the duality principle, we can rewrite the following equality for delay in distribution

\[
D_{n+1} = \max\{0, S_1, S_2, \ldots, S_n\}.
\]

\( \square \)

**Corollary 1.2.** If \( E[U_n] \geq 0 \), then for all \( c \), we have \( \Pr\{D_\infty \geq c\} = \lim_{n \to \infty} \Pr\{D_n \geq c\} = 1 \).

**Proof.** It follows from Proposition 1.1 that \( \Pr\{D_{n+1} \geq c\} \) is nondecreasing in \( n \). Hence, by MCT the limit exists and is denoted by \( \Pr\{D_\infty \geq c\} = \lim_{n \to \infty} \Pr\{D_n \geq c\} \). Therefore, by continuity of probability, we have from (1), that

\[
\Pr\{D_\infty \geq c\} = \Pr\{S_n \geq c \text{ for some } n\}. \tag{2}
\]

If \( E[U_n] = E[Y_n] - E[X_{n+1}] \) is positive, then by strong law of large numbers the random walk \( S_n \) will converge to positive infinity with probability 1. The above will also be true when \( E[U_n] = 0 \), then the random walk is recurrent.

**Remark 1.3.** Hence, we get that \( E[Y_n] < E[X_{n+1}] \) implies the existence of a stationary distribution.
Proposition 1.4 (Spitzer’s Identity). Let $M_n = \max\{0, S_1, S_2, \ldots, S_n\}$ for $n \in \mathbb{N}$, then

$$\mathbb{E}M_n = \sum_{k=1}^{n} \frac{1}{k} \mathbb{E}S_k^+.$$ 

Proof. We can decompose $M_n$ as

$$M_n = 1_{\{S_n > 0\}}M_n + 1_{\{S_n \leq 0\}}M_n.$$ 

We can rewrite first term in decomposition as,

$$1_{\{S_n > 0\}}M_n = 1_{\{S_n > 0\}} \max_{i \in [n]} S_i = 1_{\{S_n > 0\}}(X_1 + \max\{0, S_2 - S_1, \ldots, S_n - S_1\})$$

Hence, taking expectation and using exchangeability, we get

$$\mathbb{E}[M_n 1_{\{S_n > 0\}}] = \mathbb{E}[X_1 1_{\{S_n > 0\}}] + \mathbb{E}[M_{n-1} 1_{\{S_n > 0\}}].$$

Since $X_i, S_n$ has the same joint distribution for all $i$,

$$\mathbb{E}S_n^+ = \mathbb{E}[S_n 1_{\{S_n > 0\}}] = \mathbb{E}\sum_{i=1}^{n} X_i 1_{\{S_n > 0\}} = n\mathbb{E}[X_1 1_{\{S_n > 0\}}].$$

Therefore, it follows that

$$\mathbb{E}[1_{\{S_n > 0\}}M_n] = \mathbb{E}[1_{\{S_n > 0\}}M_{n-1}] + \frac{1}{n}\mathbb{E}[S_n^+].$$

Also, $S_n \leq 0$ implies that $M_n = M_{n-1}$, it follows that

$$1_{\{S_n \leq 0\}}M_n = 1_{\{S_n \leq 0\}}M_{n-1}.$$ 

Thus, we obtain the following recursion,

$$\mathbb{E}[M_n] = \mathbb{E}[M_{n-1}] + \frac{1}{n}\mathbb{E}[S_n^+].$$

Result follow from the fact that $M_1 = S_1^+$. \qed

Remark 1.5. Since $D_{n+1} = M_n$, we have $\mathbb{E}[D_{n+1}] = \mathbb{E}[M_n] = \sum_{k=1}^{n} \frac{1}{k}\mathbb{E}[S_k^+]$.

2 Martingales for Random Walks

Proposition 2.1. A random walk $S_n$ with step size $X_n \in [-M, M] \cap \mathbb{Z}$ for some finite $M$ is a recurrent DTMC iff $\mathbb{E}X = 0$.

Proof. If $\mathbb{E}X \neq 0$, the random walk is clearly transient since, it will diverge to $\pm \infty$ depending on the sign of $\mathbb{E}X$. Conversely, if $\mathbb{E}X = 0$, then $S_n$ is a martingale. Assume that the process starts in state $i$. We define

$$A = \{-M, -M + 1, \ldots, -2, -1\}, \quad A_j = \{j + 1, \ldots, j + M\}, \quad j > i.$$ 

Let $N$ denote the hitting time to $A$ or $A_j$ by random walk $S_n$. Since $N$ is a stopping time and $S_{N/n} \leq |M| + j$, by optional stopping theorem, we have

$$\mathbb{E}_i[S_N] = \mathbb{E}_i[S_0] = i.$$
Thus we have

\[ i = \mathbb{E}[S_N] \geq -M\mathbb{P}_i\{S_N \in A\} + j(1 - \mathbb{P}_i\{S_N \in A\}). \]

Rearranging this, we get a bound on probability of random walk \( S_n \) hitting \( A \) over \( A_j \) as

\[ \mathbb{P}_i\{S_n \in A \text{ for some } n\} \geq \frac{j - i}{j + M}. \]

Taking limit \( j \to \infty \), we see that for any \( i \geq 0 \), we have

\[ \mathbb{P}_i\{S_n \in A \text{ for some } n\} = 1. \]

Similarly, taking \( B = \{1, 2, \cdots, M\} \), we can show that for any \( i \geq 0 \), \( \mathbb{P}_i\{S_n \in B \text{ for some } n\} = 1 \). Result follows from combining the above two arguments to see that for any \( i \geq 0 \),

\[ \mathbb{P}_i\{S_n \in A \cup B \text{ for some } n\} = 1. \]

**Proposition 2.2.** Consider a random walk \( S_n \) with mean step size \( \mathbb{E}[X] \neq 0 \). For \( A, B > 0 \), let \( P_A \) denote the probability that the walk hits a value greater than \( A \) before it hits a value less than \( -B \). Then, for \( \theta \neq 0 \) such that \( \mathbb{E}e^{\theta X_1} = 1 \), we have

\[ P_A \approx 1 - e^{-\theta B} e^{\theta A}. \]

Approximation is an equality when step size is unity and \( A \) and \( B \) are integer valued.

**Proof.** For any \( A, B > 0 \), we can define stopping times

\[ T_A = \inf\{n \in \mathbb{N} : S_n \geq a\}, \quad T_{-B} = \inf\{n \in \mathbb{N} : S_n \leq -B\}. \]

We are interested in computing the probability

\[ P_A = \Pr\{T_A < T_{-B}\}. \]

Now let \( Z_n = e^{\theta S_n} \). We can see that \( Z_n \) is a martingale with mean 1. Define a stopping time \( N = T_A \wedge T_{-B} \). From Doob’s Theorem, \( \mathbb{E}[e^{\theta N}] = 1 \). Thus we get

\[ 1 = \mathbb{E}[e^{\theta S_N}|S_N \geq A]P_A + \mathbb{E}[e^{\theta S_N}|S_N \leq -B](1 - P_A). \]

We can obtain an approximation for \( P_A \) by neglecting the overshoots past \( A \) or \( -B \). Thus we get

\[ \mathbb{E}[e^{\theta S_N}|S_N \geq A] \approx e^{\theta A}, \quad \mathbb{E}[e^{\theta S_N}|S_N \leq -B] \approx e^{-\theta B}. \]

The result follows.