Lecture 23 : Martingales

1 Martingales

A filtration is an increasing sequence of $\sigma$-fields, with $n$th $\sigma$-field denoted by $\mathcal{F}_n$. A sequence $X = \{X_n : n \in \mathbb{N}\}$ of random variables is said to be adapted to the filtration $\{\mathcal{F}_n : n \in \mathbb{N}\}$ if $X_n \in \mathcal{F}_n$. A discrete stochastic process $\{X_n, n \in \mathbb{N}\}$ is said to be a martingale with respect to $\{\mathcal{F}_n : n \in \mathbb{N}\}$ if

i. $E[|X_n|] < \infty$,

ii. $X_n$ is adapted to $\mathcal{F}_n$,

iii. $E[X_{n+1} | \mathcal{F}_n] = X_n$, for each $n \in \mathbb{N}$.

If the equality in third condition is replaced by $\leq$ or $\geq$, then the process is called supermartingale or submartingale, respectively. For a discrete stochastic process $X = \{X_n : n \in \mathbb{N}\}$, its natural filtration is defined as

$$\mathcal{F}_n = \sigma(X_1, \ldots, X_n).$$

**Corollary 1.1.** For a martingale $X$ adapted to a filtration $\mathcal{F}$, for each $n \in \mathbb{N}$

$$E[X_n] = E[X_1].$$

**(Simple random walk).** Let $\{X_i : i \in \mathbb{N}\}$ be a sequence of independent random variables with mean $E[X_i] = 0$ and $E[|X_i|] < \infty$ for each $i \in \mathbb{N}$. Let $Z_n = \sum_{i=1}^n X_i$ and $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$ for each $n \in \mathbb{N}$. Then, $\{Z_n, n \in \mathbb{N}\}$ is a martingale with respect to the natural filtration of $X$. This follows, since $E[Z_n] = 0$ and

$$E[Z_{n+1} | \mathcal{F}_n] = E[Z_n + X_{n+1} | \mathcal{F}_n] = Z_n.$$ 

**(Product martingale).** Let $\{X_i : i \in \mathbb{N}\}$ be a sequence of independent random variables with mean $E[X_i] = 1$ and $E[|X_i|] < \infty$ for each $i \in \mathbb{N}$. Let $Z_n = \prod_{i=1}^n X_i$ and $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$. Then, $\{Z_n, n \in \mathbb{N}\}$ is a martingale with respect to the natural filtration of $X$. This follows, since $E[Z_n] = 1$ and

$$E[Z_{n+1} | \mathcal{F}_n] = E[Z_n X_{n+1} | \mathcal{F}_n] = Z_n.$$
(Branching process). Consider a population where each individual $i$ can produce an independent random number of offsprings $Z_i$ in its lifetime, given by a common distribution $P = \{P_j : j \in \mathbb{N}_0\}$ and mean $\mu = \sum_{j \in \mathbb{N}_0} j P_j$. Let $X_n$ denote the size of the $n$th generation, which is same as number of offsprings generated by $(n-1)$th generation. The discrete stochastic process $\{X_n : n \in \mathbb{N}\}$ is called a branching process. Let $X_0 = 1$ and $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$, then,

$$X_n = \sum_{i=1}^{X_{n-1}} Z_i.$$  

Conditioning on $X_{n-1}$ yields, $E[X_n] = \mu^n$ where $\mu$ is the mean number of offspring per individual. Then $\{Y_n = X_n/\mu^n : n \in \mathbb{N}\}$ is a martingale because $E[Y_0] = 1$ and

$$E[Y_{n+1} | \mathcal{F}_n] = \frac{1}{\mu^{n+1}} E\left[ \sum_{i=1}^{X_n} Z_i | \mathcal{F}_n \right] = \frac{X_n}{\mu^n} = Y_n.$$  

(Doob’s Martingale). Let $X$ be an arbitrary random variable such that $E[|X|] < \infty$, and $Y = \{Y_n : n \in \mathbb{N}\}$ be an arbitrary random sequence. Let $\mathcal{F}$ be the natural filtration associated with the stochastic process $Y$, then

$$X_n = E[X | \mathcal{F}_n]$$

is a martingale. The integrability condition can be directly verified, and

$$E[X_{n+1} | \mathcal{F}_n] = E[E[X | \mathcal{F}_{n+1}] | \mathcal{F}_n] = E[X | \mathcal{F}_n] = X_n.$$  

(Centralized Doob sequence). For any sequence of random variables $X = \{X_n : n \in \mathbb{N}\}$ and its natural filtration $\mathcal{F}$, the random variables $Z_n = X_n - E[X_n | \mathcal{F}_{n-1}]$ have zero mean, then

$$Z_n = \sum_{i=1}^{n} (X_i - E[X_i | \mathcal{F}_{i-1}])$$

is a martingale with respect to $\mathcal{F}$, provided $E[Z_n] < \infty$. To verify the same,

$$E[Z_{n+1} | \mathcal{F}_n] = E[Z_n + X_n - E[X_n | \mathcal{F}_{n-1}] | \mathcal{F}_n] = Z_n + E[X_n - E[X_n | \mathcal{F}_{n-1}]] = Z_n.$$  

Lemma 1.2. If $X = \{X_n : n \in \mathbb{N}\}$ is a martingale with respect to a filtration $\{\mathcal{F}_n : n \in \mathbb{N}\}$ and $f$ is a convex function, then $\{f(X_n) : n \in \mathbb{N}\}$ is a submartingale with respect to the same filtration.

Proof. The result is a direct consequence of Jensen’s inequality.

$$E[f(X_{n+1}) | \mathcal{F}_n] \geq f(E[X_{n+1} | \mathcal{F}_n]) = f(X_n).$$
Corollary 1.3. Let $a \in \mathbb{R}$ be a constant.

i. If $\{X_n : n \in \mathbb{N}\}$ is a submartingale, then so is $\{(X_n - a)_{+} : n \in \mathbb{N}\}$.

ii. If $\{X_n : n \in \mathbb{N}\}$ is a supermartingale, then so is $\{X_n \wedge a : n \in \mathbb{N}\}$.

1.1 Stopping Times

Consider a discrete filtration $\mathcal{F} = \{\mathcal{F}_n : n \in \mathbb{N}_0\}$. A positive integer valued, possibly infinite, random variable $N$ is said to be a random time with respect to the filtration $\mathcal{F}$, if the event $\{N = n\} \in \mathcal{F}_n$ for each $n \in \mathbb{N}$. If $\Pr\{N < \infty\} = 1$, then the random time $N$ is said to be a stopping time. A sequence $\{H_n : n \in \mathbb{N}\}$ is predictable with respect to the filtration $\mathcal{F}$, if $H_n \in \mathcal{F}_{n-1}$ for each $n \in \mathbb{N}$.

Further, we define $(H \cdot X)_n \triangleq \sum_{m=1}^{n} H_m (X_m - X_{m-1})$.

Theorem 1.4. Let $\{X_n : n \in \mathbb{N}_0\}$ be a supermartingale with respect to a filtration $\mathcal{F}$. If $H = \{H_n : n \in \mathbb{N}\}$ is predictable with respect to $\mathcal{F}$ and each $H_n$ is non-negative and bounded, then $(H \cdot X)_n$ is a supermartingale w.r.t. $\mathcal{F}$.

Proof. It follows from the definition,

$$\mathbb{E}(H \cdot X)_{n+1 | \mathcal{F}_n} = \mathbb{E}[H_{n+1} (X_{n+1} - X_n) + (H \cdot X)_n | \mathcal{F}_n] = H_{n+1} (\mathbb{E}[X_{n+1} | \mathcal{F}_n] - X_n) + (H \cdot X)_n \leq (H \cdot X)_n.$$  

Lemma 1.5. If $\{X_i : i \in \mathbb{N}\}$ is a submartingale and $T$ is a stopping time such that $\Pr\{T \leq n\} = 1$ then

$$\mathbb{E}X_1 \leq \mathbb{E}X_T \leq \mathbb{E}X_n.$$

Proof. Since $T$ is bounded, it follows from Martingale stopping theorem, that $\mathbb{E}X_T \geq \mathbb{E}X_1$. Now, since $T$ is a stopping time, we see that for $\{T = k\}$

$$\mathbb{E}[X_{n1}\{T = k\} | \mathcal{F}_T, T = k] = \mathbb{E}[X_{n1}\{T = k\} | \mathcal{F}_k] \geq X_{k1}\{T = k\} = X_T1\{T = k\}.$$

Result follows by taking expectation on both sides and summing over $k$. That is,

$$\mathbb{E}X_n = \mathbb{E} \sum_{k=1}^{n} X_{n1}\{T = k\} \geq \mathbb{E} \sum_{k=1}^{n} X_{T1}\{T = k\} = \mathbb{E}X_T.$$

Corollary 1.6. Let $T$ be a stopping time and $\{X_n : n \in \mathbb{N}\}$ be a supermartingale, then $\{X_{T\wedge n} : n \in \mathbb{N}\}$ as a supermartingale.
1.2 Stopped process

Consider a discrete stochastic process \( X = \{ X_n : n \in \mathbb{N} \} \) adapted to a discrete filtration \( \mathcal{F} \). Let \( T \) be a random time for the filtration \( \mathcal{F} \), then the **stopped process** \( \{ X_{T \wedge n} : n \in \mathbb{N} \} \) is defined as

\[
X_{T \wedge n} = X_n 1_{\{ n \leq T \}} + X_T 1_{\{ n > T \}}.
\]

**Proposition 1.7.** Let \( \{ X_n : n \in \mathbb{N} \} \) be a martingale with a discrete filtration \( \mathcal{F} \). If \( T \) is an integer random time for the filtration \( \mathcal{F} \), then the stopped process \( \{ X_{T \wedge n} \} \) is a martingale.

**Proof.** We observe that \( H = \{ 1 \{ n \leq T \} : n \in \mathbb{N} \} \) is a non-negative, predictable, and bounded sequence, since

\[
\{ n \leq T \} = \{ T > n - 1 \} = \{ T \leq n - 1 \}^c = (\cup_{i=0}^{n-1} \{ T = i \})^c = \cap_{i=1}^{n} \{ T \neq i \} \in \mathcal{F}_{n-1}.
\]

In terms of the predictable and bounded sequence \( H \), we can write the stopped process as

\[
X_{T \wedge n} = X_0 + \sum_{m=1}^{T \wedge n} (X_m - X_{m-1}) = X_0 + \sum_{m=1}^{n} 1_{\{ m \leq T \}} (X_m - X_{m-1}) = X_0 + (H \cdot X)_n.
\]

Therefore from the previous theorem we have

\[
\mathbb{E}X_{T \wedge n} = \mathbb{E}X_{T \wedge 1} = \mathbb{E}X_1.
\]

\[ \square \]

**Remark 1.8.** For any martingale \( \{ X_n : n \in \mathbb{N} \} \) w.r.t. \( \mathcal{F} \), we have \( \mathbb{E}X_{T \wedge n} = \mathbb{E}X_1 \), for all \( n \). Now assume that \( T \) is a stopping time w.r.t. \( \mathcal{F} \). It is immediate that stopped process converges almost surely to \( X_T \), i.e.

\[
\Pr \left\{ \lim_{n \in \mathbb{N}} X_{T \wedge n} = X_T \right\} = 1.
\]

We are interested in knowing under what conditions will we have convergence in mean.

**Theorem 1.9 (Martingale stopping theorem).** Let \( X \) be a martingale and \( T \) be a stopping time adapted to a discrete filtration \( \mathcal{F} \). Then, the random variable \( X_T \) is integrable and the stopped process \( X_{T \wedge n} \) converges in mean to \( X_T \), i.e.

\[
\lim_{n \in \mathbb{N}} \mathbb{E}X_{T \wedge n} = \mathbb{E}X_T = \mathbb{E}X_1,
\]

if either of the following conditions holds true.

(i) \( T \) is bounded,

(ii) \( X_{T \wedge n} \) is uniformly bounded,

(iii) \( \mathbb{E}T < \infty \), and for some real positive \( K \), we have \( \sup_{n \in \mathbb{N}} \mathbb{E}[|X_{n+1} - X_n| | \mathcal{F}_n] < K \).

**Proof.** We show this is true for all three cases.

(i) Let \( K \) be the bound on \( T \) then for all \( n \geq K \), we have \( X_{T \wedge n} = X_T \), and hence it follows that

\[
\mathbb{E}X_1 = \mathbb{E}X_{T \wedge n} = \mathbb{E}X_T, \quad \forall n \geq K.
\]

(ii) Dominated convergence theorem implies the result.
(iii) Since $T$ is integrable and

$$X_{T \wedge n} \leq |X_1| + KT,$$

we observe that $X_{T \wedge n}$ is bounded by an integrable random variable, and hence result follows from dominated convergence theorem.

\[ \square \]

**Corollary 1.10 (Wald’s Equation).** *If $T$ is a stopping time for $\{X_i : i \in \mathbb{N}\}$ iid with $E|X| < \infty$ and $ET < \infty$, then*

$$E \sum_{i=1}^{T} X_i = ETEX.$$

**Proof.** Let $\mu = E X$. Then $\{Z_n = \sum_{i=1}^{n} (X_i - \mu) : n \in \mathbb{N}\}$ is a martingale adapted to natural filtration for $X$, and hence from the Martingale stopping theorem, we have $EZ_T = EZ_1 = 0$. But

$$E[Z_T] = E \sum_{i=1}^{T} X_i - \mu ET.$$

Observe that condition (iii) for Martingale stopping theorem to hold can be directly verified. Hence the result follows. \[ \square \]