Lecture-14 : Embedded Markov Chain and Holding Times

1 State Evolution

For a Markov process on countable state space \( \mathcal{X} \) that is right continuous with left limits (rcll), we wish to know following probabilities

\[
P_{ij}(s,s+t) = P(X(s+t) = j | X(s) = i, \mathcal{F}_s), \quad s,t \geq 0.
\]

To this end, we define the sojourn time in any state, the jump times, and the jump transition probabilities.

1.1 Jump and sojourn times

The **jump times** of a right continuous stochastic process \( (X(t), \ t \geq 0) \) are defined as

\[
S_0 = 0, \quad S_n \triangleq \inf\{t > S_{n-1} : X(t) \neq X(S_{n-1})\}.
\]

The **sojourn time** of this process staying in state \( X(S_{n-1}) \) is \( T_n \triangleq (S_n - S_{n-1}) \). We denote the state of the process at \( n \)th stopping time \( S_n \) as \( X_n \triangleq X(S_n) \). It follows that \( \mathcal{F}_n = \sigma((X_n, S_n) : S_n \leq t < S_{n+1}) \). In particular, we have \( \mathcal{F}_{S_n} = \sigma((X_t, S_t) : 0 \leq t \leq n) \).

**Lemma 1.1.** Jump times \( (S_n : n \in \mathbb{N}) \) are stopping times with respect to the process \( (X(t), \ t \geq 0) \).

**Proof.** It is clear that \( \{S_n \leq t\} \) is completely determined by the history \( \mathcal{F}_t = \sigma(X(u), u \leq t) \) until time \( t \). \( \square \)

**Lemma 1.2.** For a homogeneous CTMC, each sojourn time \( T_n \) is a continuous memoryless random variable, and the sequence of sojourn times \( (T_j : j \geq n) \) is independent of the past \( \mathcal{F}_{S_{n-1}} \) conditioned on \( X_{S_{n-1}} \).

**Proof.** We observe that the sojourn time \( T_n \) equals the excess time \( Y(S_{n-1}) \) in state \( X(S_{n-1}) \) starting at time \( S_{n-1} \).

Using the strong Markov property, we can write the conditional complementary distribution of \( T_n \) given \( \mathcal{F}_{S_{n-1}} \) as

\[
\Pr(T_n > y | X_{S_{n-1}} = i, \mathcal{F}_{S_{n-1}}) = \Pr(Y(S_{n-1}) > y | X_{S_{n-1}} = i, \mathcal{F}_{S_{n-1}}) = \exp(-y \nu_i) = F_t(y), \ y \geq 0.
\]

**Corollary 1.3.** If \( X_0 = i \), then the random variable \( T_{n+1} \) has an exponential random distribution with rate \( \nu_i \).

Inverse of mean sojourn time in state \( i \) is called the **transition rate** out of state \( i \) denote by \( \nu_i = (\mathbb{E}T_i)^{-1} \).

Recall that a state \( i \) is instantaneous if \( \nu_i = \infty \), stable if \( 0 < \nu_i < \infty \), and absorbing if \( \nu_i = 0 \). Let \( N(t) \) be the counting process associated with jump times sequence \( (S_n : n \in \mathbb{N}) \). That is, the number of jumps in \( (0,t] \) is

\[
N(t) = \sum_{n \in \mathbb{N}} 1\{S_n \leq t\}.
\]

**Proposition 1.4.** For a homogeneous pure jump CTMC such that \( \inf_{i \in \mathcal{X}} \nu_i \geq \nu > 0 \) (that is all the states are stable), then the jump times are almost surely finite stopping times.

**Proof.** We observe that the jump times are sum of independent exponential random variables. Further by coupling, we can have a sequence of \( iid \) random variables \( (T_n : n \in \mathbb{N}) \), such that \( T_n \leq T_n \) and \( \mathbb{E}T_n = 1/\nu \) for each \( n \in \mathbb{N} \). Hence, we have

\[
S_n = \sum_{i=1}^{n} T_i \leq \sum_{i=1}^{\infty} \bar{T}_i \triangleq \bar{S}_n.
\]

Result follows since \( \bar{S}_n \) is the \( n \)th arrival instant of a Poisson process with rate \( \nu \). \( \square \)
1.2 Jump process

The jump process is a discrete time process \( \{X_n = X(S_n) : n \in \mathbb{N}_0\} \) derived from the continuous time stochastic process \((X(t), t \geq 0)\) by sampling at jump times. This is also sometimes referred to as the embedded DTMC of the pure jump CTMC \((X(t), t \geq 0)\). The corresponding jump transition probabilities are defined

\[
p_{ij} = P_i(S_n-1, S_n) = \Pr(X(S_n) = j | X(S_{n-1}) = i), \quad i, j \in \mathcal{X}.
\]

From the strong Markov property and the time-homogeneity of the CTMC \(X\), we see that \(P_{ij}(S_n-1, S_n) = P_{ij}(0, T_1)\).

**Lemma 1.5.** For any right continuous left limits stochastic process, the sum of jump transition probabilities \(\sum_{j \neq i} P_{ij}(S_{n-1}, S_n) = 1\) for all \(X(S_{n-1}) = i \in \mathcal{X}\).

**Proof.** It follows from law of total probability. \(\square\)

**Lemma 1.6.** For a homogeneous CTMC, the jump probability from state \(X(S_{n-1})\) to state \(X(S_n)\) depend solely on \(X(S_{n-1})\) and is independent of jump instants.

**Proof.** We can write the joint probability of \(X_n = j\) and \(T_n > x\) for any \(x \in \mathbb{R}_+\) conditioned on \(X_{n-1} = i\) and history \(\mathcal{F}_{S_{n-1}}\) for any states \(i, j \in \mathcal{X}\), using the definition of excess time \(Y(t) = S_{N(t)+1} - t\), the strong Markov property and time-homogeneity of CTMC \(X\), and memoryless property of excess time \(Y\), as

\[
P(T_n > x, X_n = j | X_{n-1} = i, \mathcal{F}_{S_{n-1}}) = P(X(x + Y(x)) = j | X(x) = i)P(Y(0) > x | X(0) = i) = P_{ij}(T_1)F(x).
\]

Result follows, since the only solution to this equation is \(P_{ij}(T_n) = P_{ij}(0)\). Hence, we can write

\[
P(T_n > x, X_n = j | X_{n-1} = i) = p_{ij}e^{-\nu_0 x}.
\]

This implies that sojourn times and jump instant probabilities are independent. \(\square\)

**Corollary 1.7.** The matrix \(P = (p_{ij} : i, j \in \mathcal{X})\) is stochastic, and if \(v_i > 0\) then \(p_{ii} = 0\).

**Proof.** Recall \(p_{ij} = p_{ij}(S_1)\). If \(v_i > 0\), then \(\lim_{u \to \infty} P(Y(0) > u | X(0) = i) = 0\), and hence \(S_1\) is finite almost surely. By definition \(X(S_1) \neq X(0) = i\), and hence \(p_{ii} = 0\). \(\square\)

**Remark.** If \(v_i = 0\), then for any \(u > 0\), we have \(P(Y(0) > u | X(0) = i) = 1\), and hence \(S_1 = \infty\) almost surely whenever \(X(0) = i\). By convention, we set \(p_i = 1\) and \(p_{ij} = 0\) for \(j \neq i\).

**Theorem 1.8.** For a pure jump CTMC \((X(t), t \geq 0)\) on state space \(\mathcal{X}\), if \(S_n\) is a proper stopping time for some \(n \in \mathbb{N}\). Then for all states \(i, j \in \mathcal{X}\) and duration \(u \geq 0\), we have

\[
P(T_{n+1} > u, X_{n+1} = j | X_0, \ldots, X_n = i, S_0, \ldots, S_n) = p_{ij}e^{-\nu_0 u}.
\]

**Proof.** Since the history of the process until stopping time \(S_n\) is given by \(\mathcal{F}_{S_n} = \sigma((X_i, S_i) : 0 \leq i \leq n)\), using strong Markov property and time-homogeneity of the CTMC \(X\), we have

\[
P(T_{n+1} > u, X_{n+1} = j | X_0, \ldots, X_n = i, S_0, \ldots, S_n) = P(T_{n+1} > u, X_{n+1} = j | \mathcal{F}_{S_n}, X_n = i) = P(S_1 > u, X_1 = j).
\]

The result follows from the previous Lemma 1.6. \(\square\)

**Corollary 1.9.** For a time-homogeneous CTMC, the transition probabilities \((p_{ij} : i, j \in \mathcal{X})\) and sojourn times \((T_n : n \in \mathbb{N})\) are independent.

**Corollary 1.10.** The jump process is a homogeneous Markov chain with countable state space \(\mathcal{X}\).

**Example 1.11 (Poisson process).** For a Poisson process with time-homogeneous rate \(\lambda\), the countable state space is \(\mathbb{N}_0\), and transition rate \(v_i = \lambda\) for each \(i \in \mathbb{N}\). This follows from the memoryless property of exponential random variables, that

\[
P(Y(u) > t | N(u) = i) = P(S_1 > t) = e^{-\lambda t}.
\]

Further, the embedded Markov chain or the jump process is given by the initial state \(N(0) = 0\) and the transition probability matrix \(P = (p_{ij} : i, j \in \mathbb{N}_0)\) where \(p_{i+1} = 1\) and \(p_{ij} = 0\) for \(j \neq i+1\). This follows from the definition of \(T_1\), since \(p_{ij} = P(N(T_1) = j | N(0) = i) = 1_{(j=i+1)}\).
1.3 Alternative construction of CTMC

Let \((X_n : n \in \mathbb{N})\) be a discrete time Markov chain with a countable state space \(\mathcal{X}\), and the transition probability matrix \(P = (p_{ij} : i, j \in \mathcal{X})\) a stochastic matrix. Further, we let \((v_i \in \mathbb{R}_+ : i \in \mathcal{X})\) be the set of transition rates such that \(p_{ii} = 0\) if \(v_i > 0\). For any initial state \(X(0) \in \mathcal{X}\), we can define a rcll piece-wise constant stochastic process \(X(t)\) inductively as

\[ X(t) = X_{n-1}, \quad t \in [S_{n-1}, S_n), \]

where \(S_0 = 0\) and \(S_n = \sum_{i=1}^{n} T_i\), where the \(n\)th transition time \(T_n\) is distributed exponentially with rate \(v_i\) if \(X_{n-1} = i\). Further, conditioned on \(X_{n-1} = i\), the transition times \((T_1, \ldots, T_n)\) and \((T_j : j \geq n)\) are mutually independent. From the definition, the process is sample-path wise right-continuous with left limits, and has countable state space. We observe that the history of the process until time \(t\) is given by \(\mathcal{F}_t = \sigma(X_n, S_n : S_n \leq t)\). We define the the number of transitions until time \(t\) by

\[ N(t) = \sum_{n \in \mathbb{N}} I_{\{S_n \leq t\}}. \]

A necessary condition for the process \(X\) to be defined on index set \(n \in \mathbb{N}_+\), is that for each \(t \geq 0\), there exists an \(n\) such that \(S_n < t < S_{n+1}\). That is, \(P\{N(t) < \infty\} = 1\) for all \(t \in \mathbb{R}_+\). This is equivalent to \(P\{S_n = \infty\} = 1\), or \(P\{S_n < \infty\} = 0\). Let \(\omega \in \{S_n < \infty\}\), then we can’t define the process for \(t > S_n\). A pure-jump CTMC \((X(t), t \geq 0)\) is called regular if \(P\{N(t) < \infty\} = 1\) for all \(t \in \mathbb{R}_+\).

**Lemma 1.12.** A homogeneous CTMC is regular if \(\sup_{i \in \mathcal{X}} v_i < v < \infty\).

**Proof.** By coupling, we can have a sequence of iid random variables \((T_n : n \in \mathbb{N})\), such that \(T_n \leq T_n\) and \(\sum T_n = v\) for each \(n \in \mathbb{N}\). Let \(m(t)\) be the associated renewal function with the sequence \(T_n\) then we can write

\[ P\{N(t) < \infty\} = \sum_{n \in \mathbb{N}_0} P\{S_n \geq t\} = 1 + m(t) \leq 1 + m(t). \]

Since the inter-renewal times have finite means, \(m(t)\) is finite, and the result follows.

**Example 1.13 (Non-regular CTMC).** For the countable state space \(\mathbb{N}\), consider the probability transition matrix \(P\) such that \(p_{i,i+1} = 1\) and the exponential holding times with rate \(v_i = i^2\) for each state \(i \in \mathbb{N}\). Clearly, \(\sup_{i \in \mathbb{N}} v_i = \infty\), and hence it is not regular.

**Lemma 1.14.** Conditioned on the process state at the beginning of an interval, the increments of the counting process \((N(t), t \geq 0)\) is independent of the past, and depends only on the duration of the increment. That is,

\[ P\{N(s,t) = k | X(s) = i, \mathcal{F}_s\} = P\{N(t-s) = k | X(0) = i\}. \]

**Proof.** From the independence of inter-transition times, we know that \(T_{N(i)+j}\) is independent of \(\mathcal{F}_s\) for \(j \geq 2\) conditioned on the process state \(X(s) = i\). Further, from the memoryless property of an exponential random variable, we have the excess time \(Y(s)\) independent of the age \(A(s)\) conditioned on the process state \(X(s) = i\). In addition, conditioned on \(X(s) = i\), the distribution of \((Y(s), T_{N(i)+2}, \ldots, T_{N(i)+k})\) is same as that of the inter-transition times \((S_1, S_2, \ldots, S_k)\) with initial state \(X(0) = i\). Hence, we can write the conditional probability of increment \(N(s,t)\) for \(t > s\), as

\[ P\{N(s,t) = k | X(s) = i, \mathcal{F}_s\} = P\{Y(s) + \sum_{i=0}^{N(s)+k} T_i \leq t | X(s) = i, \mathcal{F}_s\} = P\{N(t-s) = k\}. \]

**Proposition 1.15.** The stochastic process \((X(t), t \geq 0)\) is a time-homogeneous CTMC.

**Proof.** For state \(i, j \in \mathcal{X}\), we can write the probability of process being in state \(j\), conditioned on the past

\[ P(X(t) = j | X(s) = i, \mathcal{F}_s) = \sum_{k \in \mathbb{N}_0} P\{X(t) = j, N(s,t) = k | X(s) = i, \mathcal{F}_s\}. \]

Using the previous Lemma and definition of conditional probability, we can write for each \(k \in \mathbb{N}\),

\[ P(X(t) = j, N(s,t) = k | X(s) = i, \mathcal{F}_s) = p_{ij}^{(k)} P\{N(t-s) = k\} = P\{X(t-s) = j, N(t-s) = k\}. \]

**Theorem 1.16.** A rcll stochastic process \((X(t) \in \mathcal{X}, t \geq 0)\) defined on countable state space \(\mathcal{X}\) is a CTMC iff

1. sojourn times are independent and exponentially distributed with rate \(v_i\), where \(X(S_n-1) = i\), and
2. jump transition probabilities \(p_{ij} = P_{ij}(S_n-1, S_n)\) are independent of jump times \(S_n\) such that \(\sum_{i \neq j} p_{ij} = 1\).