Lecture-13: Continuous Time Markov Chains

1 Markov Process

For any stochastic process \((X(t), t \geq 0)\) indexed by positive reals, the history of the process until time \(t > 0\) by is the collection of all the events that can be determined by the realization of the process \(X\) until time \(t\). We denote the process history by

\[ \mathcal{F}_t = \sigma(X(u), u \leq t). \]

A real-valued stochastic process \((X(t) \in \mathbb{X}, t \geq 0)\) indexed by positive reals is a **Markov process** if it satisfies the Markov property. That is for any Borel measurable set \(A \in \mathcal{B}\), the distribution of the future states conditioned on the present, is independent of the past, and

\[ P(X(t + s) \in A | \mathcal{F}_t) = P(X(t + s) \in A | \sigma(X(s))), \quad \text{for all } s, t > 0. \]

A Markov process with countable state space \(\mathbb{X}\) is referred to as **continuous time Markov chain (CTMC)**. The Markov property for the CTMCs can be interpreted as follows. For all times \(0 < t_1 < \cdots < t_m < t\) and states \(i_1, \ldots, i_m, j \in \mathbb{X}\), we have

\[ \Pr\{X(t) = j|X(t_k) = i_k, 1 \leq k \leq m\} = \Pr\{X(t) = j|X(t_k) = i_k\}. \]

**Example 1.1 (Counting process).** Any simple counting process with independent increments is a CTMC. This implies any (possibly time-inhomogeneous) Poisson process is a CTMC. Countability of the state space is clear from the definition of the counting process. For Markov property, we observe that for \(t > s\), the increment \(N(t) - N(s)\) is independent of \(\mathcal{F}_s\). Hence for the natural filtration \(\mathcal{F}_s\),

\[ P(N(t) = j|\mathcal{F}_s) = \sum_{i \in \mathbb{N}_0} P(N(t) = j, N(s) = i|\mathcal{F}_s) = \sum_{i \in \mathbb{N}_0} 1_{\{N(s) = i\}} P(N(s, t]) = j - i = P(N(t) = j|\sigma(N(s))). \]

1.1 Transition probability kernel

We define the **transition probability** from state \(i\) at time \(s\) to state \(j\) at time \(t + s\) as

\[ P_{ij}(s, s + t) = \Pr\{X(s + t) = j|X(s) = i\}. \]

The Markov process has **homogeneous** transitions for all states \(i, j \in \mathbb{X}\) and all times \(s, t \geq 0\), if

\[ P_{ij}(s, s + t) = P_{ij}(0, t). \]

We denote the **transition probability kernel/function** at time \(t\) by \(P(t) = \{P_{ij}(t) = P_{ij}(0, t) : i, j \in \mathbb{X}\}\). We will mainly be interested in continuous time Markov chains with homogeneous jump transition probabilities. We will assume that the process \(X(t)\) is a right continuous process with left limits at each time \(t \in \mathbb{R}_+\).

**Lemma 1.2 (stochasticity).** Transition kernel \(P(t)\) at each time \(t \in \mathbb{R}_+\) is a stochastic matrix.

**Proof.** From the countable partition of the state space \(\mathbb{X}\), we get \(1 = P(X(t) \in \Omega|X(0) = i) = \sum_{j \in \mathbb{X}} P_{ij}(t). \)

**Lemma 1.3 (semigroup).** Transition kernel satisfies the semigroup property, i.e. \(P(s + t) = P(s)P(t), s, t \in \mathbb{R}_+\).

**Proof.** From the homogeneity of CTMC, we can write the \((i, j)\)th entry of \(P(s + t)\) as

\[ P_{ij}(0, s + t) = \sum_{k \in \mathbb{X}} P_{ik}(0, s) P_{kj}(s, s + t) = \sum_{k \in \mathbb{X}} P_{ik}(0, s) P_{kj}(0, t) = [P(s)P(t)]_{ij}. \]

Result follows since \(i, j \in \mathbb{X}\) were chosen arbitrarily.
Lemma 1.4 (continuity). Transition matrix $P(t)$ for a homogeneous CTMC $X$ is a continuous function of time $t \in \mathbb{R}_+$, such that $\lim_{t \to 0} P(t) = I$, the identity matrix. That is, $P_t(0) = 1$ and $P_j(0) = 0$ for all $j \neq i \in \mathcal{X}$.

Proof. It follows from continuity of probability functions and right continuity of the process at time $t = 0$. Using the semigroup property of the transition kernel, we can write $P(t + h) = P(t)P(h)$. It is easy to show that the continuity of transition kernel at $t = 0$ implies continuity of $P(t)$ at all times $t > 0$.

Since each entry of transition kernel $P(t)$ is a probability, semigroup property leads to characterization of the kernel $P(t)$ completely. For a matrix $A$ with spectral radius less than unity, we can define

$$e^A = I + \sum_{n \geq 1} \frac{A^n}{n!}$$

Lemma 1.5. We can write the transition kernel $P(t) = e^{tQ}$ for a constant matrix $e^{tQ} = P(1)$.

Proof. This follows from the semigroup property and the right continuity of transition kernel $P(t)$.

Proposition 1.6. For a time-homogeneous CTMC $(X(t), t \geq 0)$, with transition kernel $P$, for all times $0 < t_1 < \cdots < t_m$ and states $i_0, i_1, \ldots, i_m \in \mathcal{X}$, we have

$$P(X(t_k) = i_k, 1 \leq k \leq m | X(0) = i_0) = P_{i_0,i_1}(t_1)P_{i_1,i_2}(t_2 - t_1) \ldots P_{i_m,i_0}(t_m - t_{m-1}).$$

Corollary 1.7. All finite dimensional distributions of the CTMC $(X(t), t \geq 0)$ is governed by the initial distribution.

Proof. Let $v_0$ be the initial distribution of the CTMC $X$, such that $nu_0(i_0) = P(X(0) = i_0)$ for each $i_0 \in \mathcal{X}$. For all finite index sets $F \subset \mathbb{R}_+, |F| = m$ and states $(i_j \in \mathcal{X} : j \in [m])$, we have

$$P(X(t) = i_j, t_j \in F) = \sum_{i_0 \in \mathcal{X}} v_0(i_0)P_{i_0,i_1}(t_1) \ldots P_{i_{m-1},i_0}(t_m - t_{m-1}).$$

1.2 Excess time in a state

First, we define excess time for the CTMC $X$ as

$$Y(t) \triangleq \inf\{s > 0 : X(t + s) \neq X(t)\}.$$  

We observe that $Y(t)$ is the excess remaining time the process spends in state $X(t)$. For a homogeneous CTMC $X$, the distribution of excess time $Y(t)$ only depends on $X(t)$ and doesn’t depend on time $t$. Hence, we can define the following conditional complementary distribution of excess time as

$$\bar{F}_i(u) \triangleq P(Y(t) > u | X(t) = i).$$

Lemma 1.8. For a homogeneous CTMC $X$, there exists some $\nu_i > 0$, such that

$$\bar{F}_i(u) = P(Y(t) > u | X(t) = i) = e^{-u\nu_i}.$$  

Proof. It follows that $\bar{F}_i \in [0, 1]$ is non-negative, non-increasing, and right-continuous in $u$. Using the Markov property and the time-homogeneity, we can show that $\bar{F}_i$ satisfies the semigroup property. In particular,

$$\bar{F}_i(u + v) = P(Y(t) > u + v | X(t) = i) = \bar{F}_i(u)P(Y(t + u) > v | X(t + u) = i) = \bar{F}_i(u)\bar{F}_i(v).$$

The only continuous function $\bar{F}_i \in [0, 1]$ that satisfies semigroup property is an exponential function with a negative exponent.

For a CTMC $X$, a state $i \in \mathcal{X}$ is called absorbing if $\nu_i = 0$, stable if $\nu_i \in (0, \infty)$, and instantaneous if $\nu_i = \infty$. The sojourn time in an absorbing state is $\infty$, zero in an instantaneous state, and almost surely finite and non-zero in a stable state. A CTMC with no instantaneous states is called a pure jump CTMC. We will focus on pure jump CTMCs only.

Example 1.9 (Poisson process). Consider the counting process $(N(t), t \geq 0)$ for a Poisson point process with homogeneous rate $\lambda$. Using the stationary independent increment property, we have for all $u \geq 0$

$$\gamma(u) = P(N(t+u) = i | N(t) = i) = P(N(t+u) - N(t) = 0) = P(S_1 > u) = e^{-\lambda u}.$$  

A Poisson process with finite non-zero rate is a pure-jump CTMC with stable states.
1.3 Strong Markov property

A random variable \( \tau \) is a stopping time if for each \( t \in \mathbb{R}_+ \), \( \{ \tau \leq t \} \in \mathcal{F}_t \). That is, a random variable \( \tau \) is a stopping time if the event \( \{ \tau \leq t \} \) can be determined completely by the history \( \sigma(X(u), u \leq t) \). An almost surely finite stopping time \( \tau \) is called proper. A stochastic process \( X \) has strong Markov property if for any almost surely finite stopping time \( \tau \),

\[
\Pr[X(\tau + s) = j | \mathcal{F}_\tau] = \Pr[X(\tau + s) = j | X(\tau)].
\]

**Lemma 1.10.** A continuous time Markov chain \( X \) has the strong Markov property.

**Proof.** It follows from the right continuity of the CTMC process, and the fact that the map \( t \mapsto E_X f(X_{\tau+t}) \) is right-continuous for any bounded continuous function \( f : \mathcal{X} \to \mathbb{R} \). To see the right continuity of the map, we observe that

\[
E_X f(X_{\tau+t}) = \sum_{j \in \mathcal{X}} P_{ij}(s) f(j).
\]

Right-continuity of the map follows from the right continuity of the sample paths of process \( X \), right-continuity and boundedness of the kernel function, and boundedness and continuity of \( f \), and bounded convergence theorem.

**Theorem 1.11.** A pure jump CTMC \( (X(t), t \geq 0) \) satisfies the following strong Markov property. For any proper stopping time \( \tau \), finite \( m \in \mathbb{N} \), finite times \( 0 < t_1 < \cdots < t_m \), and states \( i_0, i_1, \ldots, i_m \in \mathcal{X} \), we have

\[
P(X(t_k + \tau) = i_k, 1 \leq k \leq m | X(u), u \leq \tau, X(\tau) = i_0) = P(X(t_k) = i_k, 1 \leq k \leq m | X(0) = i_0).
\]

In particular for a pure-jump time-homogeneous CTMC and proper stopping time \( \tau \), we have

\[
P(X(\tau + s) = j | X(\tau) = i, \mathcal{F}_\tau) = P_{ij}(0,s).
\]