1 Limit Theorems

Let $N_j(t)$ denote the number of transitions into state $j \in E$ up to time $t$. That is,

$$N_j(t) = \sum_{k=1}^{t} 1\{X_k = j\}.$$ 

Let $S_0 = 0$, then we can define the $n$th arrival instants of state $j$ as a stopping time

$$S_n(j) = \inf\{k > S_{n-1}(j) : X_k = j\}.$$ 

From strong Markov property it follows that $X$ is a regenerative process with regenerative sequence $S(j) = \{S_n(j) : n \in \mathbb{N}\}$. We can define the inter-renewal duration, the number of time steps to return to the state $j$ as

$$T_n(j) = S_n(j) - S_{n-1}(j).$$

If $X_0 = j$ and $j$ is recurrent, then $S(j)$ is a renewal process with the iid inter-arrival distribution,

$$P_i\{T_1(j) = k\} = f_{jj}(k), \ k \in \mathbb{N}.$$ 

Let $\mu_{jj} = \mathbb{E}_j T_1(j)$ be the mean inter-arrival time for the renewal process. Then,

$$\mu_{jj} = \begin{cases} \infty & j \text{ transient}, \\ \sum_{k \in \mathbb{N}} k f_{jj}(k) & j \text{ recurrent} \end{cases}$$

If $X_0 = i \neq j$, for some $i \leftrightarrow j$ and $j$ recurrent, then $S(j)$ is a delayed renewal process with first inter-arrival distribution

$$P_i\{T_1(j) = k\} = f_{ij}(k), \ k \in \mathbb{N}.$$ 

The associated counting process $N_j(t)$ has the inverse relationship with the renewal process $S(j)$. From the renewal theory, we have the following results.

**Proposition 1.1 (basic renewal theorem).** If $i \leftrightarrow j$, then

$$P_i\left( \lim_{n \in \mathbb{N}} \frac{N_j(n)}{n} = \frac{1}{\mu_{jj}} \right) = 1.$$ 

**Proposition 1.2 (elementary renewal theorem).** If $i \leftrightarrow j$, then

$$\lim_{n \in \mathbb{N}} \frac{\sum_{k=1}^{n} P_{ij}(k)}{n} = \lim_{n \in \mathbb{N}} \frac{\mathbb{E}_i[N_j(n)]}{n} = \frac{1}{\mu_{jj}}.$$
Proposition 1.3 (Blackwell’s theorem). If $j$ is aperiodic (i.e., $d(j) = 1$), then
\[
\lim_{n \in \mathbb{N}} P_{ij}^{(n)} = \lim_{n \in \mathbb{N}} \mathbb{E}_i [\text{# renewals at } n] = \frac{1}{\mu_{jj}}.
\]
If $j$ is periodic with period $d$, then
\[
\lim_{n \in \mathbb{N}} P_{ij}^{(nd)} = \lim_{n \in \mathbb{N}} \mathbb{E}_i [\text{# renewals at } nd] = \frac{d}{\mu_{jj}}.
\]

2 Positive and Null recurrence

A recurrent state $j$ is said to be **positive recurrent** if $\mu_{jj} < \infty$ and **null recurrent** if $\mu_{jj} = \infty$. Let
\[
\pi_j = \lim_{n \in \mathbb{N}} P_{ij}^{(nd)},
\]
where $d$ is the period of state $j$. Then $\pi_j > 0$ if and only if $j$ is positive recurrent and $\pi_j = 0$ if $j$ is null-recurrent.

Proposition 2.1. Positive recurrence and null recurrence are class properties.

An state that is aperiodic and positive recurrent is called **ergodic**. For a homogeneous Markov chain on state space $E$ with transition probability matrix $P$, a probability distribution $\{\pi_j : j \in E\}$ is said to be **stationary** if for all states $j \in E$
\[
\pi_j = \sum_{k \in E} \pi_k P_{kj}.
\]
More compactly, $\pi$ is stationary if $\pi = \pi P$.

Observe that for a Markov chain starting with its stationary distribution, then the distribution remains invariant for all times. That is, if $\pi$ is the stationary distribution, and the Markov chain has initial distribution $\nu(0) = \pi$ at time 0, then at any time $n \in \mathbb{N}$, the Markov chain has distribution $\nu(n) = \pi$. Moreover since $X_n$ has discrete states in $E$, the finite collection $(X_{n}, X_{n+1}, \ldots, X_{n+m})$ have the same joint distribution. Hence it is a stationary process, and for all $k, m \in \mathbb{N}, i \in E^k$
\[
P\{X_1 = i_1, \ldots, X_k = i_k\} = P\{X_{m+1} = i_1, \ldots, X_{m+k} = i_k\}.
\]

Theorem 2.2. An irreducible, aperiodic Markov Chain with countable state space $E$ is of one of the following types.

i) All the states are either transient or null recurrent. For all states $i, j \in E$,
\[
\lim_{n \in \mathbb{N}} P_{ij}^n = 0,
\]
and there exists no stationary distribution.

ii) All the states are positive recurrent, and hence the chain is ergodic. There exists a unique stationary distribution $\pi \in \Delta(E)$, defined for all $i, j \in E$
\[
\pi_j = \lim_{n \in \mathbb{N}} P_{ij}^n > 0.
\]

Proof. Let $\{X_n : n \in \mathbb{N}\}$ be an irreducible, aperiodic Markov chain with countable state space $E$. 

2
i) Suppose that all states are either transient or null recurrent. Note that exactly one of these will hold since there is only one communicating class. This implies that \( \mu_{jj} = \infty \) for each state \( j \in E \), and it follows from Blackwell’s theorem applied to renewals for Markov chains that for any states \( i, j \in E \)

\[
\lim_{n \in \mathbb{N}} P_{ij}^{(n)} = \frac{1}{\mu_{jj}} = 0.
\]

If there existed a stationary distribution \( \pi \in \Delta(E) \) in this case. For any step size \( n \in \mathbb{N} \) and states \( i, j \in E \), we would then have

\[
\pi_j = \sum_{i \in E} \pi_i P_{ij}^{(n)}, \quad P_{ij}^{(n)} \leq 1.
\]

We can change limits and summation using dominated convergence theorem, to get for any \( j \in E \)

\[
\pi_j = \sum_{i \in E} \pi_i \lim_{n \in \mathbb{N}} P_{ij}^{(n)} = 0.
\]

This contradicts \( \pi \) being a stationary distribution, proving the first part of the theorem.

ii) We assume that all states are positive recurrent. From the theorem hypothesis, elementary renewal theorem, and positive recurrence, we get

\[
\pi_j = \lim_{n \in \mathbb{N}} P_{ij}^{(n)} = 1/\mu_{jj} > 0.
\]

Further, for any finite set \( A \subseteq E \), we have

\[
\sum_{j \in A} P_{ij}^{(n)} = \sum_{j \in E} P_{ij}^{(n)} = 1.
\]

Taking limit \( n \in \mathbb{N} \) on both sides, we conclude that \( \sum_{j \in A} \pi_j \leq 1 \) for all \( A \) finite. Taking limit with respect to increasing sets \( A \uparrow E \), we conclude,

\[
\sum_{j \in E} \pi_j \leq 1.
\]

Further, we can write for all \( A \subseteq E \),

\[
P_{ij}^{n+1} = \sum_{k \in E} P_{ik}^{n} P_{kj} \geq \sum_{k \in A} P_{ik}^{n} P_{kj}.
\]

Applying limit \( n \in \mathbb{N} \) on both sides, we get \( \pi_j \geq \sum_{k \in A} \pi_k P_{kj} \) for all \( A \) finite. Hence, taking limits with respect to increasing sets \( A \uparrow E \), we get for all state \( j \in E \),

\[
\pi_j \geq \sum_{k \in E} \pi_k P_{kj}.
\]

Assuming that the inequality is strict for some state \( j \in E \), we can sum the inequalities over all states \( j \in E \). Since, summands are non-negative we can exchange summation orders to get

\[
\sum_{j \in E} \pi_j > \sum_{j \in E} \sum_{k \in E} \pi_k P_{kj} = \sum_{k \in E} \pi_k \sum_{j \in E} P_{kj} = \sum_{k \in E} \pi_k.
\]

This is a contradiction. Therefore, for any state \( j \in E \)

\[
\pi_j = \sum_{k \in E} \pi_k P_{kj}.
\]
Defining normalized $w_j = \frac{\pi_j}{\sum_{k \in E} \pi_k}$, we see that \( \{w_j : j \in E\} \) is a stationary distribution and so at least one stationary distribution exists. If the initial distribution of this positive recurrent Markov chain is a stationary distribution \( \{\lambda_j : j \in E\} \), then for any finite subset \( A \subseteq E \), we get
\[
\lambda_j = \Pr\{X_n = j\} = \sum_{i \in E} P^n_{ij} \lambda_i \geq \sum_{i \in A} P^n_{ij} \lambda_i.
\]
As before, we take limit \( n \in \mathbb{N} \), followed by limit of increasing subsets \( A \uparrow E \), to obtain
\[
\lambda_j \geq \sum_{i \in E} \pi_j \lambda_i = \pi_j.
\]
To show \( \lambda_j \leq \pi_j \), we use the fact that \( P^n_{ij} \leq 1 \). Let \( A \subseteq E \) be a finite subset, then
\[
\lambda_j = \sum_{i \in E} P^n_{ij} \lambda_i = \sum_{i \in A} P^n_{ij} \lambda_i + \sum_{i \not\in A} P^n_{ij} \lambda_i \leq \sum_{i \in A} P^n_{ij} \lambda_i + \sum_{i \not\in A} \lambda_i.
\]
Using our standard approach of taking limit \( n \in \mathbb{N} \), followed by \( A \uparrow E \), we obtain
\[
\lambda_j \leq \sum_{i \in E} \pi_j \lambda_i = \pi_j.
\]
\[\square\]

**Corollary 2.3.** An irreducible, aperiodic Markov chain defined on a finite state space \( E \) will be positive recurrent.

**Proof.** Suppose that the Markov chain is not positive recurrent, then
\[
\lim_{n \in \mathbb{N}} P^n_{ij} = 0.
\]
Interchanging limit and finite summation gives
\[
0 = \sum_{j \in E} \lim_{n \in \mathbb{N}} P^n_{ij} = \lim_{n \in \mathbb{N}} \sum_{j \in E} P^n_{ij} = 1.
\]
This is a contradiction. Hence the above mentioned chain is positive recurrent. \[\square\]

**Corollary 2.4.** For an irreducible and aperiodic Markov chain with stationary distribution \( \pi \) on countable state space \( E \), we have
\[
\mathbb{E}_j[T_1(j)] = \frac{1}{\pi_j}, j \in E.
\]
Further, we can define the number of visits to state \( i \) during one renewal duration \( S_1(j) \) as
\[
N_i(S_1(j)) = \sum_{k=1}^{S_1(j)} 1\{X_n = i\}.
\]

**Proposition 2.5.** For an aperiodic and irreducible Markov chain \( X \) with stationary distribution \( \pi \) on countable state space \( E \), the mean number of visits to state \( i \) in one return to state \( j \) is given
\[
\mathbb{E}_j N_i(S_1(j)) = \frac{\pi_i}{\pi_j}.
\]
Proof. Let $X_0 = j \in E$, then from renewal reward theorem for renewal sequence $S(i)$ and definition of $\pi$,

$$\lim_{n \to \infty} P_j \{X_n = i\} = \lim_{n \to \infty} \frac{\sum_{k=1}^{n} 1\{X_k = i\}}{n} = \frac{1}{E_i[T_1(i)]} = \pi_i.$$

Result follows from rewriting of the above expression for renewal sequence $S(j)$ as

$$\lim_{n \to \infty} P_j \{X_n = i\} = \lim_{n \to \infty} \frac{\sum_{k=1}^{n} 1\{X_k = i\}}{n} = \frac{\mathbb{E}_j \sum_{k=1}^{n} 1\{X_k = i\}}{E_j S_1(j)} = \frac{\mathbb{E}_j N_j(S_1(j))}{E_j T_1(j)} = \pi_j \mathbb{E}_j N_j(S_1(j)). \tag*{\Box}
$$

2.1 Ergodic theorem for Markov Chains

Proposition 2.6. Let $\{X_n : n \in \mathbb{N}_0\}$ be an irreducible, aperiodic, and positive recurrent Markov chain on countable state space $E$ with stationary distribution $\pi$. Let $f : E \to \mathbb{R}$, such that $\sum_{i \in E} |f(i)| \pi_i < \infty$, that is $f$ is integrable over $E$ with respect to $\pi$. Then for any initial distribution of $X_0$,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} f(X_n) = \sum_{i \in E} \pi_i f(i) \text{ almost surely.}$$

Proof. Fix $X_0 = i \in E$. Let $S(i)$ be sequence of successive instants at which state $i$ is visited, with $S_0(i) = 0$. For all $p \geq 0$, let $R_{p+1} = \sum_{n = S_p(i)+1}^{S_{p+1}(i)} f(X_n)$ be the net reward earned at the end of cycle $(p+1)$. Each cycle forms a renewal. By the strong Markov property, these cycles are independent. At each of these stopping times, Markov chain is in state $i \in E$. Since $\mathbb{E}_i S_1(i) = 1/\pi_i$, we get from renewal reward theorem

$$\lim_{n \to \infty} \frac{\sum_{k=1}^{n} R_k}{n} = \pi_i \mathbb{E}_i \left[ \sum_{n=1}^{S_1(i)} f(X_n) \right] = \pi_i \mathbb{E}_i \sum_{n=1}^{S_1(i)} f(j) 1\{X_n = j\}. \tag*{\Box}$$

Using dominated convergence theorem, and substituting the mean number of visits to state $j$ during successive return to state $j$, we can write

$$\lim_{n \to \infty} \frac{\sum_{k=1}^{n} R_k}{n} = \pi_i \mathbb{E}_i \sum_{j \in E} f(j) \sum_{n=1}^{S_1(i)} 1\{X_n = j\} = \pi_i \sum_{j \in E} f(j) \mathbb{E}_j N_j(S_1(i)) = \sum_{j \in E} \pi_j f(j). \tag*{\Box}$$

5