1 Renewal equation

Let \((Z_t : t \geq 0)\) be a regenerative process defined on the probability space \((\Omega, \mathcal{F}, \mathbb{P})\). Let \(F\) be the distribution of inter-renewal times, then for any measurable set \(A \in \mathcal{F}\) and \(t \geq 0\), we are interested in computing time dependent probability \(f(t) = P[Z_t \in A]\). We can write the probability of the event \(\{Z_t \in A\}\) by partitioning it into disjoint events as

\[
P[Z_t \in A] = P[Z_t \in A, S_1 > t] + P[Z_t \in A, S_1 \leq t].
\]

We define the kernel function \(K(t) = P\{S_1 > t, Z_t \in A\}\) which are typically easy to compute for any regenerative process. By the regeneration property applied at renewal instant \(S_1\), we have

\[
P[Z_t \in A, S_1 \leq t | S_1] = 1_{\{S_1 \leq t\}} E[1_{\{Z_{t-S_1}\}} | S_1] = f(t-S_1)1_{\{S_1 \leq t\}}.
\]

Hence, we have the renewal equation

\[
f(t) = K(t) + \int_0^t dF(s)f(t-s) = K + F * f.
\]

We assume that the distribution function \(F\) and the kernel \(K\) are known, and we wish to find \(f\), and characterize its asymptotic behavior.

**Example 1.1 (Age process).** Since the age process is regenerative for the associated renewal sequence, we can write the renewal equation for its distribution function as

\[
P\{A(t) \geq x\} = P\{A(t) \geq x, S_1 > t\} + \int_0^t dF(y) P\{A(t-y) \geq x\}.
\]

**Theorem 1.2.** The renewal equation has a unique solution \(f = (1 + m) * K\), where \(m(t) = \sum_{n \in \mathbb{N}} F_n(t)\) is the renewal function associated with the inter-renewal time distribution \(F\).

**Proof.** It follows from the renewal equation that \(F * ((1 + m) * K) = \sum_{n \in \mathbb{N}} F_n * K = m * K\). Hence, it is clear that \(m * K\) is a solution to the renewal equation. For uniqueness, let \(f\) be another solution, then \(h = f - K - m * K\) satisfies \(h = F * h\), and hence \(h = F_n * h\) for all \(n \in \mathbb{N}\). From finiteness of \(m(t)\), it follows that \(F_n(t) \to 0\) as \(n\) grows. Hence, \(\lim_{n \to \infty} (F_n * h)(t) = 0\) for each \(t\).

**Proposition 1.3.** Let \(Z\) be a regenerative process with state space \(V\). Then with the renewal function \(m\), measurable set \(A \in \mathcal{B}\), and the kernel function \(K(t) = P\{Z_t \in A, S_1 > t\}\), we can write for any \(t \geq 0\)

\[
P\{Z_t \in A\} = K(t) + \int_0^t dm(s)K(t-s).
\]

**Example 1.4 (Age process).** Since the age process is regenerative for the associated renewal sequence, we can write the renewal function \(K(t)\) in the renewal equation for its distribution function in terms of the complementary distribution function \(\bar{F}\) of the inter-arrival times, as \(K(t) = P\{A(t) \geq x, S_1 > t\} = 1_{\{t \geq x\}} \bar{F}(t)\). From the solution of renewal equation it follows that

\[
P\{A(t) \geq x\} = 1_{\{t \geq x\}} \bar{F}(t) + \int_0^t dm(y) 1_{\{t-y \geq x\}} \bar{F}(t-y).
\]
1.1 Delayed Regenerative Process

Theorem 1.5. Let $Z_t$ be a delayed regenerative process with the associated delayed renewal sequence $(S_n : n \in \mathbb{N})$, the renewal function $m_D$, the first arrival distribution $G$, and the common inter-arrival duration distribution $F$. For a measurable set $A \in \mathcal{B}$, we define the kernel functions $K_1(t) \equiv P[Z_t \in A, S_1 > t], K_2(t) \equiv P[Z_{t+1} \in A, t \in [0, X_2)]$, then we have

$$P[Z_t \in A] = K_1(t) + \int_0^t dm_D(y) K_2(t-y).$$

(7)

Proof. For a measurable set $A \in \mathcal{B}$, we can write the probability of the delayed regenerative process taking values in this set as disjoint sum of probability of disjoint partitions of this event as

$$P[Z_t \in A] = P[Z_t \in A, S_1 > t] + \sum_{n \in \mathbb{N}} P[Z_t \in A, N(t) = n].$$

(8)

The $n$th segment of the joint process $(N_D(t), Z(t))$ is $\zeta_n = (X_n, (Z(S_{n-1} + t) : t \in [0, X_n)))$. From the regenerative property, we know that the segments $(\zeta_n : n \in \mathbb{N})$ are independent, where $(\zeta_n : n \geq 2)$ are identically distributed. In particular, we can write

$$P[Z_t \in A, S_n \leq t < S_{n+1} | S_n] = \mathbf{1}_{(S_n < t)} P[Z_{t+S_n-S_n} \in A, t-S_n \in [0, X_2)] = \mathbf{1}_{(S_n < t)} K_2(t-S_n).$$

(9)

The result follows from the fact that $P[Z_t \in A, N(t) = n] = \mathbb{E} P[Z_t \in A, S_n \leq t < S_{n+1} | S_n].$

Example 1.6 (Age process). Age process $(A(t) = t - S_{N(t)} : t \geq 0)$ for a delayed renewal process $(S_n : n \in \mathbb{N})$ is a delayed regenerative process, since the $n$th segment is given by $\zeta_n = (X_n, (A(S_{n-1} + t) : t \geq 0, X_n))).$

Let $A = [x, \infty)$, then we can compute the kernel functions

$$K_1(t) = P[A(t) \geq x, S_1 > t] = \mathbf{1}_{(t \geq x)} \bar{G}(t), \quad K_2(t) = P[A(S_1 + t) \geq x, t \in [0, X_2)] = \mathbf{1}_{(t \geq x)} F(t).$$

Therefore, we can write the distribution of last renewal time for the delayed renewal process as

$$P[S_{N(t)} \leq x] = P[A(t) \geq t-x] = \mathbf{1}_{(x \geq 0)} \bar{G}(t) + \int_{0}^{t} dm_D(y) \mathbf{1}_{(t-y \geq x)} F(t-y).$$

(10)

Theorem 1.7 (Key Lemma). Let $S = (S_n : n \in \mathbb{N})$ be a renewal process with iid inter-renewal times $(X_n : n \in \mathbb{N})$ having common distribution function $F$, associated counting process $N(t)$, and the renewal function $m(t)$. Then,

$$\Pr\{S_{N(t)} \leq s\} = \bar{F}(t) + \int_{0}^{t} \bar{F}(t-y) dm(y), \quad t \geq s \geq 0.$$

(11)

Proof. We can see that event of time of last renewal prior to $t$ being smaller than another time $s$ can be partitioned into disjoint events corresponding to number of renewals till time $t$. Each of these disjoint events is equivalent to occurrence of $n$th renewal before time $s$ and $(n+1)$th renewal past time $t$. That is,

$$\{S_{N(t)} \leq s\} = \bigcup_{n \in \mathbb{N}_0} \{S_{N(t)} \leq s, N(t) = n\} = \bigcup_{n \in \mathbb{N}_0} \{S_n \leq s, S_{n+1} > t\}.$$

(12)

Recognizing that $S_0 = 0, S_1 = X_1$, and that $S_{n+1} = S_n + X_{n+1}$, we can write

$$\Pr\{S_{N(t)} \leq s\} = \Pr\{X_1 > t\} + \sum_{n \in \mathbb{N}} \mathbb{E}[\mathbf{1}_{\{S_n \leq t\}} | \mathbb{E}[\{X_{n+1} > t-S_n\} | S_n]].$$

(13)

We recall $F_n, n$-fold convolution of $F$, is the distribution function of $S_n$. Conditioning on $\{S_n = y\}$, we can write

$$\Pr\{S_{N(t)} \leq s\} = \bar{F}(t) + \sum_{n \in \mathbb{N}} \int_{y=0}^{t} \bar{F}(t-y) dF_n(y).$$

(14)

Using monotone convergence theorem to interchange integral and summation, and noticing that $m(y) = \sum_{n \in \mathbb{N}} F_n(y)$, the result follows.
We will not prove that if $F$ is lattice with period $d$, then
\[
\text{mated renewal function } m(T) = \text{elementary renewal theorem. To this end, note that}
\]
\[
\sum_{i=1}^{\infty} X_i = m(n) \quad n \in \mathbb{N}
\]
Taking limits on both sides of the above equation, we conclude that $g(a + b) = g(a) + g(b)$. The only increasing solution of such a $g$ is
\[
g(a) = ca, \forall a > 0,
\]
for some positive constant $c$. To show $c = \frac{1}{\mu}$, define a sequence $\{x_n, n \in \mathbb{N}\}$ in terms of $m(t)$ as
\[
x_n = m(n) - m(n-1), n \in \mathbb{N}.
\]
Note that $\sum_{i=1}^{n} x_i = m(n)$ and $\lim_{n \to \infty} x_n = g(1) = c$, hence we have
\[
\lim_{n \to \infty} \frac{\sum_{i=1}^{n} x_i}{n} = \lim_{n \to \infty} \frac{m(n)}{n} = c,
\]
where (a) follows from the fact that if a sequence \( \{x_i\} \) converges to \( c \), then the running average sequence \( a_n = \frac{1}{n} \sum_{i=1}^{n} x_i \) also converges to \( c \), as \( n \to \infty \). Therefore, we can conclude \( c = 1/\mu \) by elementary renewal theorem.

When \( F \) is lattice with period \( d \), the limit in (??) doesn’t exist. (See the following example). However, the theorem is true for lattice again by elementary renewal theorem. Indeed, since \( \frac{m(nd)}{n} \to \frac{1}{\mu} \), we can define \( x_n = m(nd) - m((n-1)d) \) and observe that \( \sum_{i=1}^{n} x_n = m(nd) \) and \( \frac{1}{n} \sum_{i=1}^{n} x_n \) converges to \( \frac{d}{\mu} \) by elementary renewal theorem.

**Example 1.10.** For a trivial lattice example where the limit in (??) does not exist, consider a renewal process with \( \Pr\{X_n = 1\} = 1 \), that is, there is a renewal at every positive integer time instant with probability 1. Then \( F \) is lattice with \( d = 1 \). Now, for \( a = 0.5 \), and \( t_n = n + (-1)^n 0.5 \), we see that \( \lim_{n \to \infty} m(t_n + a) - m(t_n) \) does not exist, and hence \( \lim_{t \to \infty} m(t + a) - m(t) \) does not exist.

**Remark 4.** In the lattice case, if the inter arrivals are strictly positive, that is, there can be no more than one renewal at each \( nd \), then we have that

\[
\lim_{n \to \infty} P[\text{renewal at } nd] = \frac{d}{\mu}.
\]