Lecture-07: Key Renewal Theorem and Applications

1 Key Renewal Theorem

Theorem 1.1 (Key Renewal Theorem). Consider a an aperiodic and recurrent renewal process with renewal function \( m(t) \), and the mean and the distribution of inter-renewal times being denoted by \( \mu \) and \( F \) respectively. For any directly Riemann integrable function \( h \in D \), we have

\[
\lim_{t \to \infty} \int_0^t h(t-x)dm(x) = \frac{1}{\mu} \int_0^\infty h(t)dt.
\]

If \( F \) is lattice with period \( d \) and \( \sum_{k \in \mathbb{N}_0} h(t+kd) \) converges, then

\[
\lim_{n \to \infty} \int_0^\infty h(t+nd-x)dm(x) = \frac{d}{\mu} \sum_{n \in \mathbb{N}_0} h(t+kd).
\]

Proposition 1.2 (Equivalence). Blackwell’s theorem and key renewal theorem are equivalent.

Proof. Let’s assume key renewal theorem is true. We select \( h \) as a simple function with value unity on interval \([0,a]\) and zero elsewhere. That is,

\[
h(x) = 1_{\{x \in [0,a]\}}.
\]

It is easy to see that this function is directly Riemann integrable. Conversely, it follows from Blackwell theorem that

\[
\lim_{t \to \infty} \frac{dm(t)}{dt} = \lim_{a \to 0} \frac{m(t+a) - m(t)}{a} = \frac{1}{\mu},
\]

where in \((a)\) we can exchange the order of limits under certain regularity conditions. We defer the formal proof for a later stage. \(\square\)

Key renewal theorem is very useful in computing the limiting value of some function \( g(t) \), probability or expectation of an event at an arbitrary time \( t \), for a renewal process. This value is computed by conditioning on the time of last renewal prior to time \( t \).

2 Alternating renewal processes

Alternating renewal processes form an important class of renewal processes, and model many interesting applications. We find one application of key renewal theorem in this section.

Let \( \{(Z_n, Y_n), \ n \in \mathbb{N}\} \) be an iid random process, where \( Y_n \) and \( Z_n \) are not necessarily independent. A renewal process where each inter-renewal time \( T_n \) consist of on time \( Z_n \) followed by off time \( Y_n \), is called an alternating renewal process. We denote the distributions for on, off, and renewal periods by \( H, G, \) and \( F \) respectively. To see that the alternating renewal process is indeed a renewal process, it needs to be established that \( \{T_n : n \in \mathbb{N}\} \) is an iid sequence. However, this follows trivially from the fact that \( \{f(Y_n, Z_n) : n \in \mathbb{N}\} \) is an iid sequence whenever \( \{(Z_n, Y_n), \ n \in \mathbb{N}\} \) is an iid sequence. Let \( f(a,b) = a + b \) to see that \( \{T_n = Y_n + Z_n : n \in \mathbb{N}\} \) is an iid sequence.

For the renewal process with nth inter-renewal time \( T_n \) for each \( n \in \mathbb{N} \), the nth renewal instant is denoted by \( S_n = \sum_{i=1}^n T_n \). We can define a stochastic process \( \{W(t) \in \{0,1\}, t \geq 0\} \) that takes values 1 and 0, when the renewal process is in on and off state respectively. In particular, we can write

\[
W(t) = 1\{A(t) < Y_{N(t)+1}\}.
\]

It is easy to see that \( W \) is a regenerative process with regenerative sequence \( S \). We denote the probability of the process being on at time \( t \) by

\[
P(t) = \Pr\{W(t) = 1\}.\]
**Theorem 2.1 (on probability).** Let \( m \) be the renewal function associated with the renewal process \( \{ S_n : n \in \mathbb{N} \} \) with a non-lattice inter-renewal duration distribution \( F \). If \( \mathbb{E}[Z_n + Y_n] < \infty \), then

\[
P(t) = \tilde{H}(t) + \int_0^t \tilde{H}(t-y)dm(y).
\]

**Proof.** We recall that \( P(t) = P\{W(t) = 1\} \) and compute the kernel

\[
P\{W(t) = 1, T_1 > t\} = P\{Z_1 > t, T_1 > t\} = P\{Z_1 > t\} = \tilde{H}(t).
\]

Hence, we can write the renewal equation for \( P(t) \) as

\[
P(t) = \tilde{H}(t) + (F \ast P)(t).
\]

Result follows from the solution of renewal equation. \( \square \)

**Corollary 2.2 (limiting on probability).** If \( \mathbb{E}[Z_n + Y_n] < \infty \) and \( F \) is non-lattice, then

\[
\lim_{t \to \infty} P(t) = \frac{\mathbb{E}[Z_n]}{\mathbb{E}[Y_n] + \mathbb{E}[Z_n]}.
\]

**Proof.** Since \( \tilde{H} \) is the distribution function of the non-negative random variable \( Z_n \), it follows that

\[
\lim_{t \to \infty} \tilde{H}(t) = 0, \quad \text{and} \quad \int_0^\infty \tilde{H}(t)dt = \mathbb{E}Z_n.
\]

Applying key renewal theorem to Theorem 2.1 we get the result. \( \square \)

# A Directly Riemann Integrable

For each \( \delta > 0 \) and \( n \in \mathbb{N} \), we define intervals \( I(n, \delta) = [(n-1)\delta, n\delta) \) that partition the positive axis \( \mathbb{R}_+ = [0, \infty) \). Let \( h : \mathbb{R}_+ \mapsto \mathbb{R} \) be a function bounded over finite intervals, denoting

\[
\underline{m}(h, n, \delta) = \inf\{m(u) : u \in I(n, \delta)\} \quad \text{and} \quad \overline{m}(h, n, \delta) = \sup\{h(u) : u \in I(n, \delta)\}.
\]

A function \( h : \mathbb{R}_+ \mapsto \mathbb{R} \) is directly **Riemann integrable** and denoted by \( h \in \mathcal{D} \) if the partial sums obtained by summing the infimum and supremum of \( h \), taken over intervals obtained by partitioning the positive axis, are finite and both converge to the same limit, for all finite positive interval lengths. That is,

\[
\sigma_\delta = \lim_{\delta \to 0} \delta \sum_{n \in \mathbb{N}} \underline{m}(h, n, \delta) = \lim_{\delta \to 0} \delta \sum_{n \in \mathbb{N}} \overline{m}(h, n, \delta) \triangleq \sigma_\delta.
\]

If both limits exist and are equal, then the integral value is equal to the limit.

**Remark 1.** We compare the definitions of directly Riemann integrable and Riemann integrable functions. For a finite positive \( M \), a function \( g : [0, M] \to \mathbb{R} \) is Riemann integrable if

\[
\lim_{\delta \to 0} \delta \sum_{n \leq M/\delta} \overline{m}(g, n, \delta) = \lim_{\delta \to 0} \delta \sum_{n \leq M/\delta} \underline{m}(g, n, \delta).
\]

In this case, the limit is the value of the integral. For \( h \) defined on \( \mathbb{R}_+ \),

\[
\int_{u \in \mathbb{R}_+} h(u)du = \lim_{M \to \infty} \int_0^M h(u)du,
\]

if the limit exists. For many functions, this limit may not exist.

**Remark 2.** A directly Riemann integrable function over \( \mathbb{R}_+ \) is also Riemann integrable, but the converse need not be true. For instance, consider the following Riemann integrable function

\[
h(t) = \sum_{n \in \mathbb{N}} 1 \left\{ t \in \left[n - \frac{1}{(2n)^2}, n + \frac{1}{(2n)^2}\right] \right\}
\]

is Riemann integrable, but \( \delta \sum_{n \in \mathbb{N}} \overline{m}(h, n, \delta) \) is always infinite for every \( \delta > 0 \).
**Proposition A.1.** Following are sufficient conditions for a function \( h \) to be directly Riemann integrable.

(a) If \( h \) is non-negative, continuous, and has finite support.

(b) If \( h \) is non-negative, continuous, bounded, and \( \sigma_\delta \) is bounded for some \( \delta \).

(c) If \( h \) is non-negative, monotone non-increasing, and Riemann integrable.

(d) If \( h \) is non-negative and bounded above by a directly Riemann integrable function.

**Proposition A.2 (Tail Property).** If \( h \) is non-negative, directly Riemann integrable, and has bounded integral value, then

\[
\lim_{t \to \infty} h(t) = 0.
\]

**B Chebyshev’s sum inequality**

**Lemma B.1.** Let \( f : \mathbb{R} \to \mathbb{R}^+ \) and \( g : \mathbb{R} \to \mathbb{R}^+ \) be arbitrary functions with the same monotonicity. For any random variable \( X \), functions \( f(X) \) and \( g(X) \) are positive and

\[
\mathbb{E}[f(X)g(X)] \geq \mathbb{E}[f(X)]\mathbb{E}[g(X)].
\]

**Proof.** Let \( Y \) be a random variable independent of \( X \) and with the same distribution. Then,

\[
(f(X) - f(Y))(g(X) - g(Y)) \geq 0.
\]

Taking expectation on both sides the result follows. \( \square \)