1 Regenerative processes

Definition 1.1 (Classical). A stochastic process $Z = (Z_t, t \geq 0)$ with the state space $E$ and the natural filtration $\mathcal{F}_s = (\mathcal{F}_s : s \geq 0)$, is said to be regenerative if there exists a sequence $S = (S_n : n \in \mathbb{N})$ of stopping times such that

(a) regeneration times: $S$ is a renewal process,

(b) regenerative property: for any $n, m \in \mathbb{N}$ and any bounded function $f : E^n \rightarrow \mathbb{R}$, we have

$$\mathbb{E}[f(Z_{S_n+t_1}, \ldots, Z_{S_n+t_n})|\mathcal{F}_{S_n}] = \mathbb{E}f(Z_{t_1}, \ldots, Z_{t_n}).$$

In particular, if the stochastic process $Z$ is bounded, then for $f(x) = x$ and $n = m = 1$, we have

$$\mathbb{E}[Z_{S_1+t}|\mathcal{F}_{S_1}] = EZ.$$ 

The definition says that probability law is independent of the past and shift invariant at renewal times. That is after each renewal instant, the process becomes an independent probabilistic replica of the process starting from zero.

Definition 1.2 (Modern). Let $N(t)$ be a counting process for a renewal process with renewal sequence $S = (S_n : n \in \mathbb{N})$ and inter-arrival sequence $(X_n : n \in \mathbb{N})$. Let $Z = (Z(t) : t \geq 0)$ be a stochastic process defined over the same probability space. The $n$th segment of the joint process $((N(t), Z(t)) : t \geq 0)$ is defined as the sample path in the $n$th inter-arrival duration, written

$$\zeta_n \triangleq (X_n, (Z_{S_n-1+t}) : t \in [0, X_n)).$$

The process $Z$ is regenerative over the renewal sequence $S$, if its segments $\{\zeta_n : n \geq 2\}$ are i.i.d. The process $Z$ is delayed regenerative, if the $N_d(t)$ is the counting process associated with a delayed renewal process, and the segments of the joint process are independent with $\{\zeta_n : n \geq 2\}$ being identically distributed.

Example 1.3 (Age process). Let $(N(t) : t \geq 0)$ be the renewal counting process for the renewal sequence $(S_n : n \in \mathbb{N})$, then the age at time $t$ is defined as $A(t) = t - S_{N(t)}$. Then the process $(A(t) : t \geq 0)$ is a regenerative process. To see this, we observe that the sample path of age in $n$th renewal interval is given by

$$A(S_{n-1} + t) = t, \quad t \in [0, X_n).$$

(3)

Since the segments $(X_n, (t : t \in [0, X_n]))$ are i.i.d., the result follows.

Example 1.4 (Markov chains). For a discrete time irreducible and positive recurrent homogeneous Markov chain $X = (X_n \in \mathbb{V} : n \in \mathbb{N})$ on finite state space $\mathbb{V}$, we can inductively define the recurrent times for state $j \in \mathbb{V}$ as $\tau_j^1(0) = 0$, and

$$\tau_j^n(n) = \inf\{k > \tau_j^{n-1}(n-1) : X_k = j\}.$$ 

(4)

From the strong Markov property of Markov chain $X$, it follows that $(\tau_j^n(n) : n \in \mathbb{N})$ is a delayed renewal sequence. Independence of the segments follows from the strong Markov property. In the $n$th segment of the joint process, we can write the joint distribution for $k < m, i \neq j$ as

$$P\{\zeta_n = (m, i)\} = P\{X(\tau_j^n(n-1)) = j\}P(\tau_j^n(n) - \tau_j^n(n-1) = m, X(\tau_j^n(n-1) + k) = i|X(\tau_j^n(n-1)) = j).$$

(5)

From the strong Markov property and the homogeneity, it follows that for $n \geq 2$

$$P\{\zeta_n = (m, i)\} = P_j(\tau_j^n(2) - \tau_j^n(1) = m, X(\tau_j^n(1) + k) = i) = P\{\zeta_2 = (m, i)\}. $$

(6)
A Stopping time $\sigma$-algebra

We wish to define an event space consisting information of the process till a random time $\tau$. For a countable stopping time $\tau$, what we want is something like $\sigma(X_t: t \leq \tau)$. But that doesn’t make sense, since the random time $\tau$ is a random variable itself. When $\tau$ is a stopping time, the event $\{\tau \leq t\} \in \mathcal{F}_t$. What makes sense is the set of all events whose intersection with $\{\tau \leq t\}$ belongs to the event subspace $\mathcal{F}_t$ for all $t \geq 0$.

For a stopping time $\tau: \Omega \to \mathbb{R}_+$ adapted to the filtration $\mathcal{F}_\cdot$, the stopping time $\sigma$-algebra is defined as

$$\mathcal{F}_\tau \triangleq \{A \in \mathcal{F}: A \cap \{t \leq \tau\} \in \mathcal{F}_t, \forall t \geq 0\}.$$ 

One can check that $\mathcal{F}_\tau$ is indeed a $\sigma$-algebra. Further, $\mathcal{F}_\tau$ has information up to the random time $\tau$. That is, it is a collection of measurable sets that are determined by the process till time $\tau$. Any measurable set $A \in \mathcal{F}$ can be written as $A = (A \cap \{\tau \leq t\}) \cup (A \cap \{\tau > t\})$. All such sets $A$ such that $A \cap \{\tau \leq t\} \in \mathcal{F}_t$ is a member of the stopped $\sigma$-algebra.

Lemma A.1. Let $\mathcal{F}_\cdot$ be the natural filtration associated with the process $(X_t: t \in T)$, and $\tau$ be the associated stopping time. Let $Y_t = X_{t\wedge \tau}$, that is $Y_t = X_11_{[t \leq \tau]} + X_t1_{[\tau > t]}$. Then $\mathcal{F}_\tau = \sigma(Y_s, s \leq \tau)$.

Proof. \qed

Lemma A.2. Let $\tau, \tau_1, \tau_2$ be stopping times adapted to a filtration $\mathcal{F}_\cdot$. Then, the following are true.

i. $\sigma(\tau) \subseteq \mathcal{F}_\tau$.

ii. If $\tau_1 \leq \tau_2$ almost surely, then $\mathcal{F}_{\tau_1} \subseteq \mathcal{F}_{\tau_2}$.

Proof. Let $\tau$ be a stopping time adapted to a filtration $\mathcal{F}_\cdot$. Then, for any $t \geq 0$, we have $\{\tau \leq t\} \in \mathcal{F}_t$.

i. We show that for any $s \geq 0$, the event $\{\tau \leq s\} \in \mathcal{F}_t$. This is true because for any $t \geq 0$

$$\{\tau \leq s\} \cap \{\tau \leq t\} = \{\tau \leq s \wedge t\} \in \mathcal{F}_t.$$

ii. From the hypothesis, we have $\{\tau_2 \leq t\} \subseteq \{\tau_1 \leq t\}$ almost surely. Let $A \in \mathcal{F}_{\tau_1}$ then $A \cap \{\tau_1 \leq t\} \in \mathcal{F}_t$ for all $t \geq 0$. Further, we see that $A \cap \{\tau_2 \leq t\} = A \cap \{\tau_2 \leq t\} \cap \{\tau_1 \leq t\} \in \mathcal{F}_t$ for all $t \geq 0$.

B Strong Markov property

Let $X$ be a real valued Markov process adapted to a filtration $\mathcal{F}_\tau$. Let $\tau$ be an almost surely finite stopping time with respect to this filtration, then the process $X$ is called strongly Markov if for all $x \in \mathbb{R}$ and $t > 0$, we have

$$P(\{X_{t+\tau} \leq x\}|\mathcal{F}_t) = P(\{X_{t+\tau} \leq x\}|\sigma(X_t)).$$

Lemma B.1. Let $(X_t: t \in T)$ be any Markov process adapted to filtration $(\mathcal{F}_t: t \in T)$. For any almost surely finite stopping time $\tau$ with respect to this filtration that takes only countably many values, Markov process $X$ is strongly Markov at stopping time $\tau$.

Proof. Let $I \subseteq T$ be the countable set such that $\{\tau \in I\} = \Omega$. Let $A \in \mathcal{F}_\tau$, then $A \cap \{\tau = i\} \in \mathcal{F}_i$ for all $i \in I$. Then,

$$E[1_A1_{\{X_{t+\tau} \leq x\} \cap \{\tau = i\}}] = \sum_{i \in I} E[1_A1_{\{X_{t+\tau} \leq x\} \cap \{\tau = i\} \cap \{\tau = i\} | \mathcal{F}_t}] = \sum_{i \in I} E[1_A1_{\{\tau = i\}}1_{\{X_{t+\tau} \leq x\} | \mathcal{F}_t}]$$

$$= \sum_{i \in I} E[1_A1_{\{\tau = i\}}1_{\{X_{t+\tau} \leq x\} | \sigma(X_i)}]$$

The result follows since $P(\{X_{t+\tau} \leq x\}|\sigma(X_t)) \in \mathcal{F}_\tau$. \qed

Corollary B.2. Any Markov process on countable index set $T$ is strongly Markov.

Proof. For a countable index set $T$, all associated stopping times assume at most countably many values. \qed

Corollary B.3. Let $\tau$ be an almost surely finite stopping time with respect to the natural filtration $\mathcal{F}_\cdot$ of an iid random sequence $X$. Then $(X_{\tau+1}, \ldots, X_{\tau+n})$ is independent of $\mathcal{F}_\tau$ for each $n \in \mathbb{N}$ and identically distributed to $(X_1, \ldots, X_n)$.
Theorem B.4. Let $X$ be any real-valued Markov process adapted to the filtration $\mathcal{F}_t$, with right-continuous sample paths. If the maps $t \mapsto \mathbb{E}_N f(X_t)$ is right-continuous for each bounded continuous function $f$, then $X$ is strongly Markov.

Proof. Let $f$ be bounded continuous function $f : E \to E$, and $t \geq 0$. Let $\sigma$ be an $\mathcal{F}_t$-adapted stopping time. It suffices to show that $f(X_t)$ satisfies the strong Markov property. Let $I_{t,m} = ((k - 1)2^{-m}, k2^{-m}]$, and consider a sequence of almost surely finite stopping times $(\sigma_m : m \in \mathbb{N})$ such that each $\sigma_m$ takes countable values and $\sigma_m \downarrow \sigma$.

By strong Markov property for countably valued stopping times, we have for each $A \in \mathcal{F}_{\sigma_m} \subseteq \mathcal{F}_\sigma$,

$$\mathbb{E}_\nu 1_A f(X_{\sigma_m + t}) = \mathbb{E}_\nu 1_A \mathbb{E}_{X_{\sigma_m}} f(X_t). \quad (7)$$

Applying dominated convergence theorem, taking limit as $\sigma_m \downarrow \sigma$ on both sides, we have

$$\mathbb{E}_\nu 1_A f(X_{\sigma + t}) = \mathbb{E}_\nu 1_A \mathbb{E}_{X_{\sigma}} f(X_t). \quad (8)$$

Corollary B.5. Let $N(t)$ be the counting process associated with the Poisson point process $S = (S_n : n \in \mathbb{N})$, then $N(t)$ satisfies the strong Markov property.

Proof. It suffices to check the right continuity of the map $t \mapsto \mathbb{E}_N f(N_t)$ for $s \geq t$, which holds from the stationary and independent increment property of Poisson process $N_t$. In particular, $N_t - N_i$ a Poisson random variable with mean $\lambda(s - t)$ and independent of $N_i$, and hence

$$\mathbb{E}_N f(N_t) = \mathbb{E}_{N_i} f(N_t - N_i + N_i) = \sum_{k \in \mathbb{N}_0} e^{-\lambda(s-t)} \frac{\lambda^k (s-t)^k}{k!} f(N_i + k). \quad (9)$$

The continuity of the map follows from the right continuity of $N_t$, boundedness and continuity of $f$, and bounded convergence theorem.

Corollary B.6. The standard Brownian motion $B(t)$ satisfies the strong Markov property.

Proof. It suffices to check the right continuity of the map $t \mapsto \mathbb{E}_B f(B_t)$ for $s \geq t$, which holds from the stationary and independent increment property of Brownian motion $B_t$. In particular, $B_s - B_t$, a Gaussian random variable with zero mean and variance $(s - t)$, independent of $B_t$. Therefore,

$$\mathbb{E}_B f(B_t) = \mathbb{E}_B f(B_s - B_t + B_t) = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2(s-t)}} f(B_t + x) dx. \quad (10)$$

The continuity of the map follows from the continuity of $B_t$, boundedness and continuity of $f$, and bounded convergence theorem.

Proposition B.7. Let $N(t)$ be the counting process associated with the renewal process $S = (S_n : n \in \mathbb{N})$, then $(N(S_{m+t}) - N(S_m), \ldots, N(S_{m+t+n} - N(S_m))$ is independent of $\sigma_m$ and has the same joint distribution as $(N(t_1), \ldots, N(t_n))$.

Proof. Recall that \(\{N(t) = k\} = \{S_k \leq t, S_{k+1} > t\}\), and hence we can write

$$\{N(S_{m+t}) - N(S_m) = k\} = \{S_{m+k} \leq S_{m+t}, S_{m+k+1} > S_{m+t}\}. \quad (11)$$

Since $S_{m+k} - S_m$ has same distribution as $S_k$ for all $k \geq 0$ and is independent of $\sigma_m$, we can write

$$P(\bigcap_{i=1}^{n} \{N(S_{m+t_i}) - N(S_m) = k_i\} \mid \sigma_m) = P(\bigcap_{i=1}^{n} \{S_{k_i} \leq t_i, S_{k_i+1} > t_i\}) = P(\bigcap_{i=1}^{n} \{N(t_i) = k_i\}). \quad (12)$$

\qed