1 Conditional Distribution of Arrivals

A Poisson process is not a stationary process. That is, the finite dimensional distributions are not shift invariant. This is clear from looking at the first moment, which is a function of time.

**Lemma 1.1.** For any finite time \( t > 0 \), a Poisson process is finite almost surely.

**Proof.** By strong law of large numbers, we have

\[
\lim_{n \to \infty} \frac{S_n}{n} = E[X] = \frac{1}{\lambda} \quad \text{a.s.}
\]

Fix \( t > 0 \) and let \( M = \{ \omega \in \Omega : N(t)(\omega) = \infty \} \) be a subset of the sample space. Let \( \omega \in M \), then \( \text{Sn}(\omega) \leq t \) for all \( n \in \mathbb{N} \). This implies \( \limsup_n \frac{S_n}{n} = 0 \) and \( \omega \not\in \{ \lim_n \frac{S_n}{n} = \frac{1}{\lambda} \} \). Hence, the probability measure for set \( M \) is zero. \( \Box \)

**Proposition 1.2 (Characterization 2).** Let \( \{I_i \subseteq \mathbb{R}_+: i \in [k]\} \) be a finite collection of disjoint intervals. A stationary independent increment simple point process \( \{N(t) : t \geq 0\} \), such that \( N(0) = 0 \) is Poisson process iff

\[
P\bigcap_{i=1}^{k} \{N(I_i) = n_i\} = \prod_{i=1}^{k} \left( \frac{\lambda |I_i|^n}{n!} \right) e^{-\lambda |I_i|}.
\]

**Proposition 1.3.** Let \( \{N(t) : t \geq 0\} \) be a Poisson process with \( \{I_i \subseteq \mathbb{R}_+: i \in [n]\} \) a set of finite disjoint intervals with \( I = \bigcup_{i \in [n]} I_i \), and \( \{k_i \in \mathbb{N}_0 : i \in [n]\} \) and \( k = \sum_{i \in [n]} k_i \). Then, we have

\[
P\{N(I) = k, i \in [n] | N(I) = k\} = k! \prod_{i \in [n]} \frac{1}{k_i!} \left( \frac{|I_i|^{k_i}}{|I|} \right).
\]

**Proof.** It follows from the stationary independent increment property of Poisson processes that

\[
P\{N(I) = k, i \in [n] | N(I) = k\} = \frac{P\bigcap_{i \in [n]} N(I_i) = k_i}{P\{N(I) = k\}} = \frac{\prod_{i \in [n]} P\{N(I_i) = k_i\}}{P\{N(I) = k\}}.
\]

\( \Box \)

**Proposition 1.4.** For a Poisson process \( \{N(t) : t \geq 0\} \), distribution of first arrival instant \( S_1 \) conditioned on \( \{N(t) = 1\} \) is uniform between \([0, t]\).

**Proof.** If \( N(t) = 1 \), then we know that conditional distribution of \( S_1 \) is supported on \([0, t)\). By Proposition 1.3 we see that

\[
P\{S_1 \leq s | N(t) = 1\} = P\{N(s) = 1, N(t-s) = 0 | N(t) = 1\} \{s < t\} + P\{N(1) = 1\} \{s \geq t\} = \frac{s}{t} \{s < t\} + 1 \{s \geq t\}.
\]

\( \Box \)
Proposition 1.5. For a Poisson process \( \{N(t), t \geq 0\} \), joint distribution of arrival instant \( \{S_1, \ldots, S_n\} \) conditioned on \( \{N(t) = n\} \) is identical to joint distribution of order statistics of iid uniformly distributed random variables between \([0, t]\).

Proof. Let \( \{s_0 = 0 < s_1 < s_2 < \ldots < s_n < t\} \) be a finite sequence of non-negative increasing numbers between 0 and \( t \). Then, by Proposition 1.3, we get

\[
P\{S_i \leq s_i, i \in [n] | N(t) = n\} = P\{N((0, s_i]) \geq i, i \in [n] | N(t) = n\}.
\]

Alternative proof. Let \( \{s_i \in (0, t) : i \in [n]\} \) be a sequence of increasing numbers. If we denote \( s_0 = 0 \), then we can write

\[
\bigcap_{i=1}^{n} \{S_i = s_i\} \cap \{N(t) = n\} \iff \bigcap_{i=1}^{n} \{X_i = s_i - s_{i-1}\} \cap \{X_{n+1} > t - s_n\}.
\]

Note that all the events on RHS are independent events. Therefore, it is easy to compute the joint distribution of \( \{S_1, \ldots, S_n\} \), as

\[
P\bigg[\bigcap_{i=1}^{n} \{S_i \leq s_i\} \cap \{N(t) = n\} = \int_{0}^{s_1} dt_1 \cdots \int_{0}^{s_n} dt_n \prod_{i=1}^{n} \lambda \exp(-\lambda t_i) \exp(-\lambda (t - u_n)) = \lambda^n \exp(-\lambda t) \prod_{i=1}^{n} s_i.
\]

Since \( P\{N(t) = n\} = \exp(-\lambda t)(\lambda t)^n/n! \), it follows that

\[
P\{S_1 \leq s_1, \ldots, S_n \leq s_n | N(t) = n\} = \begin{cases} n! \prod_{i=1}^{n} \frac{s_i}{i} & s < t \\ 0 & s \geq t \end{cases}
\]

Let \( U_1, \ldots, U_n \) are iid Uniform random variables in \([0, t]\). Then, the order statistics of \( U_1, \ldots, U_n \) has an identical joint distribution to \( n \) arrival instants conditioned on \( \{N(t) = n\} \). \( \square \)

2. Age and excess time

Definition 2.1. For a point process \( \{N(t), t \geq 0\} \), we can define age process \( \{A(t), t \geq 0\} \) and excess time process \( \{Y(t), t \geq 0\} \) as

\[
A(t) = t - S_{N(t)}, \quad Y(t) = S_{N(t)+1} - t.
\]

Proposition 2.2. For a Poisson process with rate \( \lambda \), the corresponding age and excess time are both exponentially distributed with rate \( \lambda \) irrespective of time \( t \).

Proof. Using stationary independent increment property of Poisson process, we can write complementary distribution of excess time process as

\[
P\{Y(t) > y\} = \sum_{n \in \mathbb{N}_0} P\{Y(t) > y; N(t) = n\} = \sum_{n \in \mathbb{N}_0} P\{N(t+y) - N(t) = 0, N(t) = n\}
\]

\[
= P\{N(y) = 0\} \sum_{n \in \mathbb{N}_0} P\{N(t) = n\} = P\{N(y) = 0\}.
\]

2
Theorem 3.1 (Sum of Independent Poissons). Let \( \{N_1(t), t \geq 0\} \) and \( \{N_2(t), t \geq 0\} \) be two independent Poisson processes with rates \( \lambda_1 \) and \( \lambda_2 \) respectively. Then, the process \( \{N(t) = N_1(t) + N_2(t)\} \) is Poisson with rate \( \lambda_1 + \lambda_2 \).

Proof. We need to show that \( \{N(t)\} \) has stationary independent increments, and

\[
P\{N(t) = n\} = \exp\left(- (\lambda_1 + \lambda_2) t\right) \frac{(\lambda_1 + \lambda_2)^n t^n}{n!}.
\]

For two disjoint intervals \( (t_1, t_2) \) and \( (t_3, t_4) \), we can see that for both processes \( N_1(t) \) and \( N_2(t) \), arrivals in \( (t_1, t_2) \) and \( (t_3, t_4) \) are independent. Therefore, \( N(t) \) has independent increment property. Similarly, we can argue about the stationary increment property of \( \{N(t)\} \). Further, we can write

\[
\{N(t) = n\} = \bigcup_{k=0}^{n} \{\{N_1(t) = k\} \cap \{N_2(t) = n-k\}\}.
\]

Since \( N_1(t) \) and \( N_2(t) \) are independent, we can write

\[
P\{N(t) = n\} = \sum_{k=0}^{n} \exp(-\lambda_1 t) \frac{(\lambda_1 t)^k}{k!} \exp(-\lambda_2 t) \frac{(\lambda_2 t)^{n-k}}{(n-k)!} = \frac{\exp(- (\lambda_1 + \lambda_2) t)}{n!} \sum_{k=0}^{n} \binom{n}{k} \frac{\lambda_1^k \lambda_2^{n-k}}{k!}.
\]

Result follows by recognizing that summand is just binomial expansion of \( [(\lambda_1 + \lambda_2) t]^n \). □

Remark 3.2. If independence condition is removed, the statement is not true.

Theorem 3.3 (Independent Splitting). Let \( \{N(t), t \geq 0\} \) be a Poisson arrival process. Each arrival can be randomly assigned to either arrival type 1 or 2, with probability \( p \) and \( (1-p) \) respectively, independent of previous assignments. Arrival processes of type 1 and 2 are denoted by \( N_1(t) \) and \( N_2(t) \) respectively. Then, \( \{N_1(t), t \geq 0\} \) and \( \{N_2(t), t \geq 0\} \) are mutually independent Poisson processes with rates \( \lambda p \) and \( \lambda (1-p) \) respectively.

Proof. To show that \( N_1(t), t \geq 0 \) is a Poisson process with rate \( \lambda p \), we show that it is stationary independent increment process with the distribution

\[
P\{N_1(t) = n\} = \frac{(p \lambda t)^n}{n!} e^{-\lambda p t}.
\]
The stationary, independent increment property of the probabilistically filtered processes \( \{N_1(t), t \geq 0\} \) and \( \{N_2(t), t \geq 0\} \) can be understood and argued out from the example given in the figure. Notice that

\[
\{N_1(t) = k\} = \bigcup_{n=k}^{\infty} \{N(t) = n, N_1(t) = k\}.
\]

Further notice that conditioned on \( \{N(t) = n\} \), probability of event \( \{N_1(t) = k\} \) is merely probability of selecting \( k \) arrivals out of \( n \), each with independent probability \( p \). Therefore,

\[
P(N_1(t) = k) = \exp(-\lambda t) \sum_{n=k}^{\infty} \frac{\lambda^n}{n!} \left(\frac{1}{k}\right)^n (1-p)^{n-k},
\]

\[
= \exp(-\lambda t) \frac{(\lambda p)^k}{k!} \sum_{n=k}^{\infty} \frac{\lambda^n}{n!} (1-p)^{n-k}.
\]

Recognizing that infinite sum in RHS adds up \( \exp(\lambda(1-p)t) \), the result follows. We can find the distribution of \( N_2(t) \) by similar arguments. We will show that events \( \{N_1(t) = n_1\} \) and \( \{N_2(t) = n_2\} \) are
independent. To this end, we see that

$$\{N_1(t) = n_1, N_2(t) = n_2\} = \{N(t) = n_1 + n_2, N_1(t) = n_1\}.$$  

Using their distribution for $N_1(t), N_2(t)$, and conditional distribution of $N_1(t)$ on $N(t)$, we can show that

$$P\{N_1(t) = n_1, N_2(t) = n_2\} = \exp(-\lambda t) \frac{(\lambda t)^{n_1 + n_2}}{(n_1 + n_2)!} \left(\frac{n_1 + n_2}{n_1}\right) p^{n_1} (1-p)^{n_2};$$

$$= P\{N_1(t) = n_1\} P\{N_2(t) = n_2\}.$$

In general, we need to show finite dimensional distributions factorize. That is, we need to show that for measurable sets $A_1, \ldots, A_n : j \in [m]$, we have

$$P\left( \bigcap_{i=1}^n \{N_1(t_i) \in A_i\} \bigcap_{j=1}^m \{N_2(s_j) \in B_j\} \right) = P\left( \bigcap_{i=1}^n \{N_1(t_i) \in A_i\} \right) P\left( \bigcap_{j=1}^m \{N_2(s_j) \in B_j\} \right).$$

$\square$