Lecture 04: Properties of Poisson Process

1 Characterizations of Poisson process

It is clear that \( t \) partitions \( X_{N(t)+1} \) in two parts such that \( X_{N(t)+1} = A(t) + Y(t) \) as seen in Figure ?? for the case when \( N(s) = n \).

![Figure 1: Stationary and independent increment property of Poisson process.](image)

**Proposition 1.1.** A Poisson process \( \{N(t), t \geq 0\} \) is simple counting process with stationary independent increments.

**Proof.** It is clear that Poisson process is a simple counting process. To show that \( N(t) \) has stationary and independent increments, it suffices to show that \( N(t) - N(s) \) is independent of \( N(s) \) and the distribution of increment \( N(t) - N(s) \) is identical to that of \( N(t-s) \). This follows from the fact that we can use induction to show stationary and independent increment property for any finite disjoint time-intervals.

We can write the distribution of \( N(t) - N(s) \) given \( N(s) \) in terms of the following events involving inter-arrival times and excess times as

\[
P\{N(t) - N(s) \geq m | N(s) = n\} = P\{Y(s) + S_{n+m} - S_{n+1} \leq t - s | S_n + A(s) = s\}.
\]

Further, we see that independent increment holds only if inter-arrival time is exponential. Since, \( \{X_i : i \geq n+2\} \cup \{Y(s)\} \) are independent of \( \{X_i : i \leq n\} \cup A(s) \), we have \( N(t) - N(s) \) independent of \( N(s) \). Further, since \( Y(s) \) has same distribution as \( X_{n+1} \), we get \( N(t) - N(s) \) having same distribution as \( N(t-s) \). By induction, we can extend this result to \( (N(t_n) - N(t_{n-1}), ..., N(s)) \). \( \square \)
Theorem 1.2 (Characterization 1). A simple counting process with stationary and independent increment is a Poisson process with parameter $\lambda$ when

$$\lim_{t \to 0} \frac{P\{N(t) = 1\}}{t} = \lambda, \quad \text{and} \quad \lim_{t \to 0} \frac{P\{N(t) \geq 2\}}{t} = 0.$$

Proof. It suffices to show that first inter-arrival times $X_1$ is exponentially distributed with parameter $\lambda$. Notice that, the probability $P_0(t)$ of no arrivals in a time duration $[0,t)$ satisfies the semi-group property. That is,

$$P_0(t + s) = P\{N(t + s) - N(t) = 0, N(t) = 0\} = P_0(t)P_0(s).$$

Using the conditions in the theorem, the result follows. \hfill \Box

Proposition 1.3 (Characterization 2). Let $\{I_i \subseteq \mathbb{R}_+ : i \in [k]\}$ be a finite collection of disjoint intervals. A stationary and independent increment simple counting process $\{N(t), t \geq 0\}$ with $N(0) = 0$ is Poisson process iff

$$P \left( \bigcap_{i=1}^{k} \{N(I_i) = n_i\} \right) = \prod_{i=1}^{k} \left( \frac{\lambda |I_i|^{n_i}}{n_i!} e^{-\lambda |I_i|} \right).$$

Proof. It is clear that Poisson process satisfies the above conditions. Further, since $P\{N(t) = 0\} = e^{-\lambda t}$, it follows that the counting process with stationary and independent increment is Poisson with rate $\lambda$. \hfill \Box

Proposition 1.4. Let $\{N(t), t \geq 0\}$ be a Poisson process with $\{I_i \subseteq \mathbb{R}_+ : i \in [n]\}$ a set of finite disjoint intervals with $I = \bigcup_{i \in [n]} I_i$, and $\{k_i \in \mathbb{N}_0 : i \in [n]\}$ and $k = \sum_{i \in [n]} k_i$. Then, we have

$$P\{N(I_i) = k_i, i \in [n]|N(I) = k\} = k! \prod_{i \in [n]} \frac{1}{k_i!} \left( \frac{|I_i|}{|I|} \right)^{k_i}.$$

Proof. It follows from the stationary and independent increment property of Poisson processes that

$$P\{N(I_i) = k_i, i \in [n]|N(I) = k\} = \frac{P\left( \bigcap_{i \in [n]} \{N(I_i) = k_i\} \right)}{P\{N(I) = k\}} = \frac{\prod_{i \in [n]} P\{N(I_i) = k_i\}}{P\{N(I) = k\}}.$$

1.1 Conditional distribution of arrivals

Proposition 1.5. For a Poisson process $\{N(t), t \geq 0\}$, distribution of first arrival instant $S_1$ conditioned on $\{N(t) = 1\}$ is uniform between $[0,t]$.

Proof. If $N(t) = 1$, then we know that conditional distribution of $S_1$ is supported on $[0,t)$. By Proposition 1.4, we see that

$$P\{S_1 \leq s|N(t) = 1\} = P\{N(s) = 1, N(t-s) = 0|N(t) = 1\} 1\{s < t\} + 1\{s \geq t\} = \frac{s}{t} 1\{s < t\} + 1\{s \geq t\} = s.$$

Proposition 1.6. For a Poisson process $\{N(t), t \geq 0\}$, joint distribution of arrival instant $\{S_1, \ldots, S_n\}$ conditioned on $\{N(t) = n\}$ is identical to joint distribution of order statistics of $n$ iid uniformly distributed random variables between $[0,t]$.
Proof. Let \( \{s_i \in (0, t) : i \in [n]\} \) be a sequence of increasing numbers. If we denote \( s_0 = 0 \), then we can write
\[
\bigcap_{i=1}^{n} \{S_i = s_i\} \cap \{N(t) = n\} \iff \bigcap_{i=1}^{n} \{X_i = s_i - s_{i-1}\} \cap \{X_{n+1} > t - s_n\}.
\]
Note that all the events on RHS are independent. Hence, it is easy to compute the joint distribution of \( \{S_1, \ldots, S_n\} \) as
\[
P\left(\bigcap_{i=1}^{n} \{S_i \leq s_i\} \cap \{N(t) = n\}\right) = \int_{0}^{s_1} du_1 \cdots \int_{0}^{s_n} du_n \prod_{i=1}^{n} \lambda \exp(-\lambda(u_i - u_{i-1})) \exp(-\lambda(t - u_n))
\]
\[
= \lambda^n \exp(-\lambda t) \prod_{i=1}^{n} s_i.
\]
Since \( P\{N(t) = n\} = \exp(-\lambda t) (\lambda t)^n / n! \), it follows that
\[
P\{S_1 \leq s_1, \ldots, S_n \leq s_n | N(t) = n\} = \begin{cases} n! \prod_{i=1}^{n} \frac{s_i}{t} & s < t \\ 0 & s \geq t. \end{cases}
\]
Let \( U_1, \ldots, U_n \) be iid uniform random variables in \([0, t]\). Then, the order statistics of \( U_1, \ldots, U_n \) has an identical joint distribution to \( n \) arrival instants conditioned on \( \{N(t) = n\}\).

\[\square\]

2 Superposition and decomposition of Poisson processes

Theorem 2.1 (Sum of Independent Poissons). Let \( \{N_1(t), t \geq 0\} \) and \( \{N_2(t), t \geq 0\} \) be two independent Poisson processes with rates \( \lambda_1 \) and \( \lambda_2 \) respectively. Then, the process \( N(t) = N_1(t) + N_2(t) \) is Poisson with rate \( \lambda_1 + \lambda_2 \).

Proof. We need to show that \( \{N(t)\} \) has stationary independent increments, and
\[
P\{N(t) = n\} = \exp(- (\lambda_1 + \lambda_2) t) \frac{\lambda_1^n \lambda_2^n}{n!}.
\]
For two disjoint interval \((t_1, t_2)\) and \((t_3, t_4)\), we can see that for both processes \(N_1(t)\) and \(N_2(t)\), arrivals in \((t_1, t_2)\) and \((t_3, t_4)\) are independent. Therefore, \(N(t)\) has independent increment property. Similarly, we can argue about the stationary increment property of \( \{N(t)\} \). Further, we can write
\[
\{N(t) = n\} = \bigcup_{k=0}^{n} \left\{ \{N_1(t) = k\} \cap \{N_2(t) = n-k\} \right\}.
\]
Since \(N_1(t)\) and \(N_2(t)\) are independent, we can write
\[
P\{N(t) = n\} = \sum_{k=0}^{n} \exp(-\lambda_1 t) \frac{(\lambda_1 t)^k}{k!} \exp(-\lambda_2 t) \frac{(\lambda_2 t)^{n-k}}{(n-k)!}
\]
\[
= \frac{\exp(-(\lambda_1 + \lambda_2) t)}{n!} \sum_{k=0}^{n} \binom{n}{k} (\lambda_1 t)^k (\lambda_2 t)^{n-k}.
\]
Result follows by recognizing that summand is just binomial expansion of \((\lambda_1 + \lambda_2) t^n\).

Remark 2.2. If independence condition is removed, the statement is not true.
Figure 2: Splitting a Poisson process into two independent Poisson processes.

**Theorem 2.3 (Independent Splitting).** Let \( \{N(t), t \geq 0\} \) be a Poisson arrival process. Each arrival can be randomly assigned to either arrival type 1 or 2, with probability \( p \) and \( (1 - p) \) respectively, independent of previous assignments. Arrival processes of type 1 and 2 are denoted by \( N_1(t) \) and \( N_2(t) \) respectively. Then, \( \{N_1(t), t \geq 0\} \) and \( \{N_2(t), t \geq 0\} \) are mutually independent Poisson processes with rates \( \lambda p \) and \( \lambda (1 - p) \) respectively.

**Proof.** To show that \( N_1(t), t \geq 0 \) is a Poisson process with rate \( \lambda p \), we show that it is stationary independent increment process with the distribution

\[
P\{N_1(t) = n\} = \frac{(p \lambda t)^n}{n!} e^{-\lambda pt}.
\]

The stationary, independent increment property of the probabilistically filtered processes \( \{N_1(t), t \geq 0\} \) and \( \{N_2(t), t \geq 0\} \) can be understood and argued out from the example given in the figure. Notice that

\[
\{N_1(t) = k\} = \bigcup_{n=k}^{\infty} \{N(t) = n, N_1(t) = k\}.
\]
Further notice that conditioned on \( \{ N(t) = n \} \), probability of event \( \{ N_1(t) = k \} \) is merely probability of selecting \( k \) arrivals out of \( n \), each with independent probability \( p \). Therefore,

\[
P\{ N_1(t) = k \} = \exp(-\lambda t) \sum_{n=0}^{\infty} \left( \frac{(\lambda t)^n}{n!} \right) p^n (1-p)^{n-k},
\]

\[
= \exp(-\lambda t) \left( \frac{(\lambda p)^k}{k!} \right) \sum_{n=k}^{\infty} \left( \frac{(\lambda (1-p)t)^{n-k}}{(n-k)!} \right).
\]

Recognizing that infinite sum in RHS adds up \( \exp(\lambda (1-p)t) \), the result follows. We can find the distribution of \( N_2(t) \) by similar arguments. We will show that events \( \{ N_1(t) = n_1 \} \) and \( \{ N_2(t) = n_2 \} \) are independent. To this end, we see that

\[
\{ N_1(t) = n_1, N_2(t) = n_2 \} = \{ N(t) = n_1 + n_2, N_1(t) = n_1 \}.
\]

Using their distribution for \( N_1(t), N_2(t) \), and conditional distribution of \( N_1(t) \) on \( N(t) \), we can show that

\[
P\{ N_1(t) = n_1, N_2(t) = n_2 \} = \exp(-\lambda t) \left( \frac{(\lambda t)^{n_1+n_2}}{(n_1+n_2)!} \right) \left( \frac{n_1+n_2}{n_1} \right) p^{n_1} (1-p)^{n_2},
\]

\[
= P\{ N_1(t) = n_1 \} P\{ N_2(t) = n_2 \}.
\]

In general, we need to show finite dimensional distributions factorize. That is, we need to show that for measurable sets \( A_1, \ldots, A_m : j \in [m] \), we have

\[
P\left( \bigcap_{j=1}^{m} \{ N_1(t_i) \in A_i \} \cap \bigcap_{j=1}^{m} \{ N_2(s_j) \in B_j \} \right) = P\left( \bigcap_{j=1}^{m} \{ N_1(t_i) \in A_i \} \right) P\left( \bigcap_{j=1}^{m} \{ N_2(s_j) \in B_j \} \right).
\]

\( \square \)

A  Order statistics

For any \( n \) length sequence \( a \in \mathbb{R}^n \), the order statistics is a permutation \( \sigma : [n] \to [n] \) such that

\[
a_{\sigma(1)} \leq a_{\sigma(2)} \leq \cdots \leq a_{\sigma(n)}.
\]

For, \( k \in [n] \), we call \( a_{\sigma(k)} \) as the \( k \)th order statistic of the sequence \( a \). In particular, first order statistic is the minimum, and the \( n \)th order statistic is the maximum of a \( n \) length sequence.

Lemma A.1. Let \( X = (X_1, X_2, \ldots, X_n) \) be an \( n \) length sequence of iid random variables with common distribution and density functions \( F \) and \( f \) respectively. Then, the joint density of order statistics of sequence \( X \) for \( x \in \mathbb{R}^n \) is

\[
f_{x_{\sigma}}(x) = n! \prod_{i=1}^{n} f(x_i).
\]

Lemma A.2. Let \( X = (X_1, X_2, \ldots, X_n) \) be an \( n \) length sequence of iid random variables with common distribution and density functions \( F \) and \( f \) respectively. Then, the density function of \( k \)th order statistic of sequence \( X \) for \( x \in \mathbb{R} \) is

\[
f_{x_{\sigma(k)}}(x) = \binom{n}{k} F(x)^{k-1} F(x)^{n-k} f(x).
\]