1 Simple point processes

A point process is a collection $\Phi = (S_n \in \mathbb{R}^d : n \in \mathbb{N})$ of randomly distributed points, such that $\lim_{n \to \infty} |S_n| = \infty$. A point process is simple if the points are distinct. Let $N(0) = 0$ and denote the number of points in a measurable set $A \in \mathcal{B}(\mathbb{R}^d)$ by

$$N(A) = \sum_{n \in \mathbb{N}} 1 \{ S_n \in A \}. \quad (1)$$

Then $(N(A) : A \in \mathcal{B}(\mathbb{R}^d))$ is called a counting process for the simple point process $\Phi$. In $\mathbb{R}_+$, one can order these points as an increasing sequence $(S_n : n \in \mathbb{N})$, and denote number of points in half-open interval by

$$N(t) \triangleq N(0, t] = \sum_{n \in \mathbb{N}} 1 \{ S_n \in (0, t] \} = \sum_{n \in \mathbb{N}} 1 \{ S_n \leq t \}. \quad (2)$$

**Lemma 1.1.** A stochastic process $(N(t) : t \geq 0)$ is a counting process if

i. $N(0) = 0$, and

ii. for each $\omega \in \Omega$, the map $t \mapsto N(t)$ is non-decreasing, integer valued, and right continuous.

**Lemma 1.2.** A counting process has finitely many jumps in a finite interval $(0, t]$.

The points of discontinuities of the counting process $N(t)$ correspond to the arrival instants of the corresponding point process $\{ S_n : n \in \mathbb{N} \}$, where the random variable $S_n$ is called the $n$th arrival instant.

$$S_0 = 0, \quad S_n = \inf \{ t \geq 0 : N(t) \geq n \}, \quad n \in \mathbb{N}.$$ 

The inter arrival time between $(n-1)$th and $n$th arrival is denoted by $X_n = S_n - S_{n-1}$. For a simple point process, we have $P[X_n = 0] = P[X_n \leq 0] = 0$. General point processes in higher dimension don’t have any inter-arrival time interpretation. Let $\mathcal{F}_* = (\mathcal{F}_t : s \geq 0)$ be the natural filtration associated with the counting process $N(t)$, that is $\mathcal{F}_t = \sigma(N(t), t \geq 0)$. Then $(S_n : n \in \mathbb{N})$ are sequences of stopping times.

![Sample path of a simple counting process](image)

**Lemma 1.3 (Inverse processes).** Simple counting process $(N(t), t \geq 0)$ and the corresponding arrival process $(S_n : n \in \mathbb{N})$ are inverse processes. That is,

$$\{ S_n \leq t \} = \{ N(t) \geq n \}. \quad (3)$$

**Proof.** Let $\omega \in \{ S_n \leq t \}$. Since $N$ is a non-decreasing process, we have $N(t) \geq N(S_n) = n$. Conversely, let $\omega \in \{ N(t) \geq n \}$, then it follows from definition that $S_n \leq t$. \qed
2 Renewal processes

We will consider inter-arrival times \((X_i : i \in \mathbb{N})\) to be a sequence of non-negative iid random variables with a common distribution \(F\), finite mean \(\mu\), and \(F(0) < 1\). We interpret \(X_n\) as the time between \((n-1)^{th}\) and the \(n^{th}\) renewal event. Let \(S_n\) denote the time of \(n^{th}\) renewal instant and assume \(S_0 = 0\). Then, we have

\[
S_n = \sum_{i=1}^{n} X_i, \quad n \in \mathbb{N}.
\]

Second condition on inter-arrival times implies non-degenerate renewal process. If \(F(0)\) is equal to 1 then it is a trivial process. The associated counting process \((N(t), t \geq 0)\) with iid general inter-arrival times is called a renewal process, written as

\[
N(t) = \sup\{n \in \mathbb{N}_0 : S_n \leq t\} = \sum_{n\in\mathbb{N}} I\{S_n \leq t\}.
\]

**Example 2.1 (Random walk).** Random walk \(S\) on \(\mathbb{R}^d\) with iid non-negative step-sizes \((X_n : n \in \mathbb{N})\) is a renewal process.

**Example 2.2 (Markov chain).** Let \(X\) be a discrete time homogeneous Markov chain \(X\) with state space \(V\). For \(X_0 = i \in V\) and \(\tau_i^+(0) = 0\), let the recurrent times be defined inductively as

\[
\tau_i(n)^+ = \inf\{k > \tau_i^+(n-1) : X_k = i\}.
\]

It follows from the strong Markov property of the process \(X\), that \((\tau_i^+(n) : n \in \mathbb{N})\) is a renewal sequence.

Many times in practice, we have a delayed start to a renewal process. That is, the arrival process has independent inter-arrival times \((X_n : n \in \mathbb{N})\), where the common distribution for \(X_n\) for \(n \geq 2\) is \(F\), and the distribution of first inter-arrival time is \(G\). The associated counting process is called a delayed renewal process and denoted by \((N_D(t) : t \geq 0)\). Let \(S_0 = 0\) and \(n^{th}\) arrival instant \(S_n = \sum_{i=1}^{n} X_i\). Then, then following inverse relationship holds between counting and arrival process,

\[
N_D(t) = \sup\{n \in \mathbb{N} : S_n \leq t\}.
\]

**Example 2.3 (Markov chain).** Let \(X\) be a discrete time homogeneous Markov chain \(X\) with state space \(V\). For \(X_0 = i \in V\) and for \(j \neq i\) let \(\tau_j^+(0) = 0\), let the recurrent times be defined inductively as

\[
\tau_j(n)^+ = \inf\{k > \tau_j^+(n-1) : X_k = j\}.
\]

It follows from the strong Markov property of the process \(X\), that \((\tau_j^+(n) : n \in \mathbb{N})\) is a delayed renewal sequence.

A renewal process \(S\) is said to be **recurrent** if the inter-renewal time \(X_n\) is finite almost surely for every \(n \in \mathbb{N}\), the process is called **transient** otherwise. The process is said to be **periodic** with period \(d\) if the inter-renewal times \((X_n : n \in \mathbb{N})\) take values in a discrete set \(\{nd : n \in \mathbb{N}_0\}\) and \(d\) is the largest such number. Otherwise, if there is no such \(d > 0\), then the process is said to be aperiodic. If the inter-arrival times is a periodic random variable, then the associated distribution function \(F\) is called **lattice**.

**Lemma 2.4 (Finiteness).** For a renewal process with positive \(E X_n > 0\), the number of renewals \(N(t)\) in the time duration \((0, t]\) is a.s. finite for all \(t > 0\).

**Proof.** We are interested in knowing how many renewals occur per unit time. From strong law of large numbers, we know that

\[
P\left\{\lim_{n \to \infty} \frac{S_n}{n} = \mu\right\} = 1.
\]
Since \( \mu > 0 \), we must have \( S_n \) growing arbitrarily large as \( n \) increases. Thus, \( S_n \) can be finite for at most finitely many \( n \). Indeed for any finite \( t \), we have the the following set inclusion

\[
\bigcap_{n \in \mathbb{N}} \{ N(t) \geq n \} = \bigcap_{n \in \mathbb{N}} \{ S_n \leq t \} \subseteq \bigcap_{n \in \mathbb{N}} \left\{ \frac{S_n}{n} \leq \frac{t}{n} \right\} \subseteq \left\{ \limsup_{n \in \mathbb{N}} \frac{S_n}{n} = 0 \right\}.
\]  

(9)

Since, \( \left\{ \limsup_{n \in \mathbb{N}} \frac{S_n}{n} = 0 \right\} \subseteq \left\{ \lim_{n \in \mathbb{N}} \frac{S_n}{n} = \mu \right\}^c \), it follows that \( P\{N(t) = \infty\} = 0 \) for any finite \( t \). The result follows and \( N(t) = \max\{n \in \mathbb{N}_0 : S_n \leq t \} \).

**Corollary 2.5.** For delayed renewal process with positive \( \mathbb{E}X_n > 0 \) for \( n \geq 2 \) and finite mean \( \mathbb{E}X_1 \), the number of renewals \( N_D(t) \) in the time duration \((0, t]\) is a.s. finite for all \( t > 0 \).

### 2.1 Distribution functions

For two functions \( F, G : \mathbb{R}_+ \to \mathbb{R} \), the convolution of \( F, G \) is denoted by \( F * G \), and defined as the function

\[
(F * G)(t) \triangleq \int_0^t F(t-y)dG(y).
\]

(10)

The distribution function of \( n \)-th renewal instant \( S_n \) is denoted by \( F_n(t) \triangleq P\{S_n \leq t\} \) for all \( t \in \mathbb{R} \).

**Lemma 2.6.** The distribution function \( F_n \) is computed inductively as \( F_n = F_{n-1} * F \), where \( F_1 = F \).

**Proof.** It follows from induction over sum of iid random variables. The base case is clear from definition. We assume the induction hypothesis to be true for \( n - 1 \). Then, we can write

\[
F_n(t) = \mathbb{E}1_{\{S_{n-1} + X_n \leq t\}} = \mathbb{E}[\mathbb{E}[1_{\{S_{n-1} + X_n \leq t\}}] | X_n] = \mathbb{E}[F_{n-1}(t-X_n)].
\]

(11)

The last equality from the independence of \( S_{n-1} \) and \( X_n \).

**Corollary 2.7.** The distribution function of \( n \)-th arrival instant \( S_n \) for delayed renewal process is \( G * F_{n-1} \).

**Lemma 2.8.** Counting process \( N(t) \) assumes non-negative integer values with distribution

\[
P\{N(t) = n\} = P\{S_n \leq t\} - P\{S_{n+1} \leq t\} = F_n(t) - F_{n+1}(t).
\]

(12)

**Proof.** It follows from the inverse relationship between renewal instants and the renewal process (??).

**Corollary 2.9.** The distribution function of counting process \( N_D(t) \) for the delayed renewal process is

\[
P\{N_D(t) = n\} = P\{S_n \leq t\} - P\{S_{n+1} \leq t\} = (G * F_{n-1})(t) - (G * F_n)(t).
\]

(13)

### 2.2 Renewal function

Mean of the counting process \( N(t) \) is called the **renewal function** denoted by \( m(t) = \mathbb{E}[N(t)] \).

**Proposition 2.10.** Renewal function can be expressed in terms of distribution of renewal instants as

\[
m(t) = \sum_{n \in \mathbb{N}} F_n(t).
\]

**Proof.** Using the inverse relationship between counting process and the arrival instants, we can write

\[
m(t) = \mathbb{E}[N(t)] = \sum_{n \in \mathbb{N}} P\{N(t) > n\} = \sum_{n \in \mathbb{N}} P\{S_n \leq t\} = \sum_{n \in \mathbb{N}} F_n(t).
\]

We can exchange integrals and summations since the integrand is positive using monotone convergence theorem.

**Corollary 2.11.** The renewal function for the delayed renewal process \( N_D(t) \) is \( m_D = G * (1 + m) \).

**Proof.** We can write the renewal function for the delayed renewal process as

\[
m_D(t) = \mathbb{E}N_D(t) = \sum_{n \in \mathbb{N}} (G * F_{n-1})(t) = G(t) + (G * m)(t).
\]

(14)
Proposition 2.12. For renewal process with $\mathbb{E}X_n > 0$, the renewal function is bounded for all finite times.

Proof. Since we assumed that $P\{X_n = 0\} < 1$, it follows from continuity of probabilities that there exists $\alpha > 0$, such that $P\{X_n \geq \alpha\} = \beta > 0$. We can define bivariate random variables

$$\tilde{X}_n = \alpha \mathbb{1}_{\{X_n \geq \alpha\}} \leq X_n.$$  \hspace{1cm} (15)

Note that since $X_i$’s are iid, so are $\tilde{X}_i$’s. Each $\tilde{X}_i$ takes values in $\{0, \alpha\}$ with probabilities $1 - \beta$ and $\beta$ respectively. Let $\tilde{N}(t)$ denote the renewal process with inter-arrival distribution $\tilde{F}_n$, with arrivals at integer multiples of $\alpha$. Then for all sample paths, we have

$$N(t) = \sum_{n \in \mathbb{N}} 1_{\{\sum_{i=1}^{n} \tilde{X}_i \leq t\}} = \sum_{n \in \mathbb{N}} 1_{\{\sum_{i=1}^{n} \tilde{X}_i \leq t\}} = \tilde{N}(t).$$  \hspace{1cm} (16)

Hence, it follows that $\mathbb{E}N(t) \leq \mathbb{E}\tilde{N}(t)$, and we will show that $\mathbb{E}\tilde{N}(t)$ is finite. We can write the joint distribution of number of arrivals at each arrival instant $t\alpha$, as

$$P\{\tilde{N}(0) = n_1, \tilde{N}(\alpha) = n_2\} = P\left(\bigcap_{i=1}^{n_1} \{X_i \leq \alpha\} \bigcap \{X_{n_1+1} \geq \alpha, X_{n_2+1} \geq \alpha\} \bigcap \{X_{n_1+i} < \alpha\}\right) = (1 - \beta)^{n_1} \beta (1 - \beta)^{n_2 - 1} \beta.$$  \hspace{1cm} (17)

It follows that the number of arrivals is independent at each arrival instant $k\alpha$ and geometrically distributed with mean $1/\beta$ and $(1 - \beta)/\beta$ for $k \in \mathbb{N}$ and $k = 0$ respectively. Thus, for all $t \geq 0$,

$$\mathbb{E}N(t) \leq \mathbb{E}\tilde{N}(t) \leq \frac{\lfloor \frac{t}{\alpha} \rfloor}{\beta} \leq \frac{t}{\alpha} + 1 \frac{1}{\beta} < \infty.$$  \hspace{1cm} (18)

\hfill \Box

Corollary 2.13. For delay renewal process with $\mathbb{E}X_n > 0$, the renewal function is bounded at all finite times.

2.3 Moment generating function

Laplace transform for a function $f(t)$ is denoted by $\tilde{f}(s)$ and defined as

$$\tilde{f}(s) = \int_{\mathbb{R}_+} e^{-st} df(t).$$  \hspace{1cm} (19)

Lemma 2.14. Let $F$ be a distribution function for a random variable $X$ with the Laplace transform as $\tilde{F}(s)$, then the Laplace transform of $F^2 = F \ast F$ is given by $\tilde{F}^2(s) = (\tilde{F}(s))^2$.

Proof. It follows from the change of variables for the integration below,

$$\tilde{F}^2(s) = \int_0^\infty dt e^{-st} \int_0^t F(t-u) dF(u) = \int_0^\infty e^{-st} F(u) \int_u^\infty dF(t-u)e^{-s(t-u)} = (\tilde{F}(s))^2.$$  \hspace{1cm} (20)

\hfill \Box

Corollary 2.15. Laplace transform of a renewal function $m$, is given by $\frac{1}{1-F(s)}$ for $s$ such that $\mathbb{R}\{\tilde{F}(s)\} < 1$.

Lemma 2.16. Let the Laplace transforms for the renewal function $m_D$, the first inter-arrival time distribution $G$, and the subsequent inter-arrival distribution $F$, be denoted by $\tilde{m}_D(s), \tilde{G}(s), \tilde{F}(s)$ respectively, for a delayed renewal process. Then,

$$\tilde{m}_D(s) = \frac{\tilde{G}(s)}{1-F(s)}, \quad \mathbb{R}\{\tilde{F}(s)\} < 1.$$  \hspace{1cm} (21)