1 Simple point processes

A simple point process is a collection of distinct points
\[ \Phi = \{ S_n \in \mathbb{R}^d : n \in \mathbb{N} \}, \]
such that \(|S_n| \to \infty\) as \(n \to \infty\). In \(\mathbb{R}_+\), one can order these points \(\{ S_n : n \in \mathbb{N} \}\). Let \(N(\emptyset) = 0\) and denote the number of points in a set \(A \subseteq \mathbb{R}^d\) by
\[ N(A) = \sum_{n \in \mathbb{N}} 1\{ S_n \in A \}. \]
Then \(\{N(A) : A \in \mathcal{F}\}\) is called a counting process for the simple point process \(\Phi\). A counting process is simple if the jump size is unity.

1. Point processes can model arrivals at classrooms, banks, hospital, supermarket, traffic intersections, airports etc.
2. Point processes can model location of nodes in a network, such as cellular networks, sensor networks, etc.

We can simplify this definition for \(d = 1\). A stochastic process \(\{N(t), t \geq 0\}\) is a counting process if
1. \(N(0) = 0\), and
2. for each \(\omega \in \Omega\), the map \(t \mapsto N(t)\) is non-decreasing, integer valued, and right continuous.

Lemma 1.1. A counting process has finitely many jumps in a finite interval \([0, t)\).

The points of discontinuity correspond to the arrival instants of the point process \(N(t)\). The \(n\)th arrival instant is a random variable denoted \(S_n\), such that
\[ S_0 = 0, \quad S_n = \inf\{ t \geq 0 : N(t) \geq n \}, \quad n \in \mathbb{N}. \]
The inter arrival time between \((n - 1)\)th and \(n\)th arrival is denoted by \(X_n\) and written as
\[ X_n = S_n - S_{n-1}. \]

For a simple point process, we have
\[ P\{X_n = 0\} = P\{X_n \leq 0\} = 0. \]

General point processes in higher dimension don’t have any inter-arrival time interpretation.
Figure 1: Sample path of a simple counting process.

**Lemma 1.2.** Simple counting process \(\{N(t), t \geq 0\}\) and arrival process \(\{S_n : n \in \mathbb{N}\}\) are inverse processes. That is,
\[
\{S_n \leq t\} = \{N(t) \geq n\}.
\]

**Proof.** Let \(\omega \in \{S_n \leq t\}\), then \(N(S_n) = n\) by definition. Since \(N\) is a non-decreasing process, we have \(N(t) \geq N(S_n) = n\). Conversely, let \(\omega \in \{N(t) \geq n\}\), then it follows from definition that \(S_n \leq t\). \(\square\)

**Corollary 1.3.** The following identity is true.
\[
\{S_n \leq t, S_{n+1} > t\} = \{N(t) = n\}.
\]

**Proof.** It is easy to see that
\[
\{S_{n+1} > t\} = \{S_{n+1} \leq t\}^c = \{N(t) \geq n + 1\}^c = \{N(t) < n + 1\}.
\]
Hence, the result follows by writing
\[
\{N(t) = n\} = \{N(t) \geq n, N(t) < n + 1\} = \{S_n \leq t, S_{n+1} > t\}.
\]

**Lemma 1.4.** Let \(F_n(x)\) be the distribution function for \(S_n\), then
\[
P_n(t) \triangleq P\{N(t) = n\} = F_n(t) - F_{n+1}(t).
\]

**Proof.** It suffices to observe that following is a union of disjoint events,
\[
\{S_n \leq t\} = \{S_n \leq t, S_{n+1} > t\} \cup \{S_n \leq t, S_{n+1} \leq t\}.
\]

\(\square\)
1.1 Stationary and independent increments

For an interval $I = (s, t]$, the number of arrivals in the interval $I$ is defined as $N(I) = N(t) - N(s)$. Consider an arbitrary collection of mutually exclusive intervals $\{I_j : j \in [n]\}$, time index $t \geq 0$, and set of positive integers $\{k_j \in \mathbb{N}_0 : j \in [n]\}$. A counting process $\{N(t), t \geq 0\}$ has **stationary increments** if

$$P\{N(I_j) = k_j, j \in [n]\} = P\{N(t + I_j) = k_j, j \in [n]\}.$$ 

A counting process $\{N(t), t \geq 0\}$ has **independent increments** if

$$P\{N(I_j) = k_j, j \in [n]\} = \prod_{j \in [n]} P\{N(I_j) = k_j\}.$$ 

**Lemma 1.5.** An arrival process $\{S_n, n \in \mathbb{N}_0\}$ has stationary and independent increments iff the sequence of inter-arrival times $\{X_n : n \in \mathbb{N}\}$ are iid random variables.

**Proof.** We first suppose that $\{X_n : n \in \mathbb{N}\}$ is a sequence of iid random variables. Then $S_{n+m} - S_m$ has the same distribution as $S_n$ and is independent of $(X_1, \ldots, X_m)$. Conversely, we suppose that $\{S_n : n \in \mathbb{N}_0\}$ has stationary and independent increments. Then, $\{X_n : n \in \mathbb{N}\}$ is a sequence of iid random variables by looking at $X_n = S_n - S_{n-1}$. \hfill \Box

**Lemma 1.6.** If a simple counting process $\{N(t), t \geq 0\}$ has stationary and independent increments then the sequence of inter-arrival times $\{X_n : n \in \mathbb{N}\}$ are iid random variables.

**Proof.** First, we notice that from inverse relationship, we have

$$\{X_n > y\} = \{N(S_{n-1}) \leq N(S_{n-1} + y) < N(S_n)\} = \{N(S_{n-1} + y) - N(S_{n-1}) = 0\}.$$ 

To show that each inter-arrival time is identically distributed, we utilize the stationarity of the increments of the counting process $N(t)$, to observe

$$P\{S_n - S_{n-1} > y\} = \int_0^\infty P\{N(y) = 0\}dF_{n-1}(t) = P\{N(y) = 0\} = P\{X_1 > y\}.$$ 

To show that inter-arrival times are independent, it suffices to show that $X_n$ is independent of $S_{n-1}$. Since the increments of the counting process $N(t)$ are independent, we see that

$$P\{S_{n-1} \leq x, X_n > y\} = \int_0^x P\{N(y + t) - N(t) = 0 | S_{n-1} = t\}dF_{n-1}(t)$$

$$= \int_0^x P\{N(y + t) - N(t) = 0 | N(t) = n - 1, N(s) < n - 1, s < t\}dF_{n-1}(t)$$

$$= P\{X_n > y\} F_{n-1}(x).$$ 

\hfill \Box

2 Poisson process

A simple counting process $\{N(t), t \geq 0\}$ is called a **Poisson process** with a finite positive rate $\lambda$, if the inter-arrival times $\{X_n : n \in \mathbb{N}\}$ are iid random variables with an exponential distribution of rate $\lambda$. That is, it has a distribution function $F$, such that

$$F(x) = P\{X_1 \leq x\} = \begin{cases} 
1 - e^{-\lambda x}, & x \geq 0 \\
0, & \text{else}.
\end{cases}$$

For many proofs regarding Poisson processes, we partition the sample space with the disjoint events $\{N(t) = n\}$ for $n \in \mathbb{N}_0$. We need the following lemma that enables us to do that.
Lemma 2.1. For any finite time $t > 0$, a Poisson process is finite almost surely.

Proof. By strong law of large numbers, we have

$$\lim_{n \to \infty} \frac{S_n}{n} = E[X_1] = \frac{1}{\lambda} \text{ a.s.}$$

Fix $t > 0$ and we define a sample space subset $M = \{ \omega \in \Omega : N(t)(\omega) = \infty \}$. For any $\omega \in M$, we have $S_n(\omega) \leq t$ for all $n \in \mathbb{N}$. This implies $\limsup_n \frac{S_n}{n} = 0$ and $\omega \notin \{ \lim_n \frac{S_n}{n} = \frac{1}{\lambda} \}$. Hence, the probability measure for set $M$ is zero. $\square$

2.1 Memoryless distribution

A random variable $X$ with continuous support on $\mathbb{R}_+$ is called memoryless if

$$P\{X > t + s | X > t\} = P\{X > s\} \forall t, s \in \mathbb{R}_+.$$

Proposition 2.2. The unique memoryless distribution function with continuous support on $\mathbb{R}_+$ is the exponential distribution.

Proof. Let $X$ be a random variable with a memoryless distribution function $F : \mathbb{R}_+ \to [0, 1]$. It follows that $\bar{F}(t) \triangleq 1 - F(t)$ satisfies the semi-group property

$$\bar{F}(t+s) = \bar{F}(t) \bar{F}(s).$$

Since $\bar{F}(x) = P\{X > x\}$ is non-increasing in $x \in \mathbb{R}_+$, we have $\bar{F}(x) = e^{\theta x}$, for some $\theta < 0$ from Lemma A.1. $\square$

2.2 Distribution functions

Lemma 2.3. Moment generating function of arrival times $S_n$ is

$$\mathbb{E}[e^{\theta S_n}] = \begin{cases} \frac{\lambda^n}{(\lambda - \theta)^n}, & \theta < \lambda \\ \infty, & \theta \geq \lambda. \end{cases}$$

Lemma 2.4. Distribution function of $S_n$ is given by

$$F_n(t) \triangleq P\{S_n \leq t\} = 1 - e^{-\lambda t} \sum_{k=0}^{n-1} \frac{(\lambda t)^k}{k!}.$$

Theorem 2.5. Density function of $S_n$ is Gamma distributed with parameters $n$ and $\lambda$. That is,

$$f_n(s) = \frac{\lambda^n (\lambda s)^{n-1}}{(n-1)!} e^{-\lambda s}.$$

Theorem 2.6. For each $t > 0$, the distribution of Poisson process $N(t)$ with parameter $\lambda$ is given by

$$P\{N(t) = n\} = e^{-\lambda t} \frac{(\lambda t)^n}{n!}.$$

Further, $\mathbb{E}[N(t)] = \lambda t$, explaining the rate parameter $\lambda$ for Poisson process.
**Proof.** Result follows from density of \( S_n \) and recognizing that

\[
P_n(t) = F_n(t) - F_{n+1}(t).
\]

\[\square\]

**Corollary 2.7.** Distribution of arrival times \( S_n \) is

\[
F_n(t) = \sum_{j \geq n} P_j(t), \quad \sum_{n \in \mathbb{N}} F_n(t) = \mathbb{E}N(t).
\]

**Proof.** First result follows from the telescopic sum and the second from the following observation.

\[
\sum_{n \in \mathbb{N}} F_n(t) = \mathbb{E} \sum_{n \in \mathbb{N}} 1\{N(t) \geq n\} = \sum_{n \in \mathbb{N}} P\{N(t) \geq n\} = \mathbb{E}N(t).
\]

\[\square\]

A Poisson process is not a stationary process. That is, the finite dimensional distributions are not shift invariant. This is clear from looking at the first moment \( \mathbb{E}N(t) = \lambda t \), which is linearly increasing in time.

### 2.3 Age and excess time

At any time \( t \), the instant of last and next arrivals are \( S_{N(t)} \) and \( S_{N(t)+1} \) respectively. **Age** of a counting process defined as age from the last arrival, and the **excess** is defined as remaining time till next arrival,

\[
A(t) = t - S_{N(t)} \quad \text{and} \quad Y(t) = S_{N(t)+1} - t.
\]

**Lemma 2.8.** Age and residual processes for a Poisson process are independent and the corresponding residual process has distribution same as inter-arrival distribution

**Proof.** We first find the distribution of age \( A(s) \) and excess time \( Y(s) \) individually. Using stationary increment property of the counting process \( N(t) \), we can write

\[
P\{A(s) > x\} = \sum_{n \in \mathbb{N}_0} P\{N(s) - N(s-x) = 0, N(s) = n\} = \sum_{n \in \mathbb{N}_0} P\{N(s) = 0, N(s-x) = n\} = F_0(x),
\]

\[
P\{Y(s) > y\} = \sum_{n \in \mathbb{N}_0} P\{N(s+y) - N(s) = 0, N(s) = n\} = \sum_{n \in \mathbb{N}_0} P\{N(y) = 0, N(s) = n\} = F_0(y).
\]

Since the counting process \( N(t) \) has stationary and independent increments, we can write the joint probability as

\[
P\{A(s) > x, Y(s) > y\} = \sum_{n \in \mathbb{N}_0} P\{N(s+y) - N(s-x) = 0, N(s-x) = n\} = \sum_{n \in \mathbb{N}_0} P\{N(y) = 0, N(s-x) = n\}
\]

\[
= P\{N(y) = 0\} = P\{N(y) = 0\}P\{N(y) = 0\} = P_0(x)P_0(y).
\]

Therefore, \( Y(s) \) is independent of \( A(s) \) and they both have the same exponential distribution as \( X_{n+1} \). The memoryless property of exponential distribution is crucially used. \(\square\)
A Functions with semigroup property

Lemma A.1. A unique non-negative right continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfying equation

$$f(t + s) = f(t)f(s), \text{ for all } t, s \in \mathbb{R}$$

is $f(t) = e^{\theta t}$, where $\theta = \log f(1)$.

Proof. Clearly, we have $f(0) = f^2(0)$. Since $f$ is non-negative, it means $f(0) = 1$. By definition of $\theta$ and induction for $m, n \in \mathbb{Z}^+$, we see that

$$f(m) = f(1)^m = e^{\theta m}, \quad e^{\theta} = f(1) = f(1/n)^n.$$  

Let $q \in \mathbb{Q}$, then it can be written as $m/n, n \neq 0$ for some $m, n \in \mathbb{Z}^+$. Hence, it is clear that for all $q \in \mathbb{Q}^+$, we have $f(q) = e^{\theta q}$, either unity or zero. Note, that $f$ is a right continuous function and is non-negative. Now, we can show that $f$ is exponential for any real positive $t$ by taking a sequence of rational numbers $\{t_n\}$ decreasing to $t$. From right continuity of $f$, we obtain

$$f(t) = \lim_{t_n \uparrow t} f(t_n) = \lim_{t_n \uparrow t} e^{\theta t_n} = e^{\theta t}.$$

\qed