1 Simple point processes

Definition 1.1. A stochastic process \( \{N(t), t \geq 0\} \) is a point process if

1. \( N(0) = 0 \), and

2. for each \( \omega \in \Omega \), the map \( t \mapsto N(t) \) is non-decreasing, integer valued, and right continuous.

Definition 1.2. A simple point process is a point process of jump size 1.

![Sample path of a simple point process.](image)

Definition 1.3. We can define a random variable \( S_n \) as the time of \( n^{th} \) discontinuity, written

\[
S_0 = 0, \quad S_n = \inf\{t \geq 0 : N(t) = n\}, n \in \mathbb{N}.
\]

The points of discontinuity corresponds to the arrival instants of the point process.

Lemma 1.4. Simple point process \( \{N(t), t \geq 0\} \) and arrival process \( \{S_n : n \in \mathbb{N}\} \) are inverse processes. That is,

\[
\{S_n \leq t\} = \{N(t) \geq n\}.
\]

Proof. Let \( \omega \in \{S_n \leq t\} \), then \( N(S_n) = n \) by definition. Since \( N \) is a non-decreasing process, we have \( N(t) \geq N(S_n) = n \). Conversely, let \( \omega \in \{N(t) \geq n\} \), then it follows from definition that \( S_n \leq t \).
**Corollary 1.5.** The following identity is true.

\[ \{S_n \leq t, S_{n+1} > t\} = \{N(t) = n\}. \]

**Proof.** It is easy to see that

\[ \{S_{n+1} > t\} = (\{S_{n+1} \leq t\})^c = (N(t) \geq n + 1)^c = (N(t) < n + 1). \]

Hence, the result follows by writing

\[ \{N(t) = n\} = \{N(t) \geq n, N(t) < n + 1\} = \{S_n \leq t, S_{n+1} > t\}. \]

\[ \square \]

**Lemma 1.6.** Let \( F_n(x) \) be the distribution function for \( S_n \), then

\[ P_n(t) \triangleq \Pr\{N(t) = n\} = F_n(t) - F_{n+1}(t). \]

**Proof.** It suffices to observe that following is a union of disjoint events,

\[ \{S_n \leq t, S_{n+1} > t\} \cup \{S_n \leq t, S_{n+1} \leq t\} = \{S_n \leq t\}. \]

\[ \square \]

**Definition 1.7.** The inter arrival time between \((n-1)^{th}\) and \(n^{th}\) arrival is denoted by \( X_n \) and written as

\[ X_n = S_n - S_{n-1}. \]

**Remark 1.8.** For a simple point process, we have

\[ \Pr\{X_n = 0\} = \Pr\{X_n \leq 0\} = 0. \]

Notice that, generalized point processes in higher dimension don’t have any inter-arrival time interpretation.

**Definition 1.9.** For an interval \( I = (s, t] \), the number of arrivals in the interval \( I \) is defined as \( N(I) = N(t) - N(s) \).

**Definition 1.10.** A point process \( \{N(t), t \geq 0\} \) is called stationary increment point process, if for any collection of mutually exclusive intervals \( \{I_j : j \in [n]\} \) and any \( t \geq 0 \)

\[ \Pr\cap_{j \in [n]} \{N(I_j) = k_j\} = \Pr\cap_{j \in [n]} \{N(t + I_j)\}. \]

**Definition 1.11.** A point process \( \{N(t), t \geq 0\} \) is called stationary independent increment process, if it has stationary increments and the increments are independent random variables.

**Remark 1.12.** For any collection of points \( t_0 = 0 < t_1 < \ldots < t_n \), we can define mutually exclusive intervals \( I_j = (t_{j-1}, t_j] \) for \( j \in [n] \). Then, a stationary independent increment point process is the one that has the joint probability distribution, a product of marginals and each marginal depends solely on the interval length \( |I_j| = t_j - t_{j-1} \). That is,

\[ \Pr\cap_{j \in [n]} \{N(I_j) = n_j\} = \prod_{j \in [n]} \Pr\{N(|I_j|) = n_j\}. \]
Lemma 1.13. An arrival process \( \{S_n, n \in \mathbb{N}_0\} \) has stationary and independent increments iff the sequence of inter-arrival times \( \{X_n : n \in \mathbb{N}\} \) are iid random variables.

Proof. We first suppose that \( \{X_n : n \in \mathbb{N}\} \) is a sequence of iid random variables. Then \( S_{n+m} - S_m \) has the same distribution as \( S_n \) and is independent of \( (X_1, \ldots, X_m) \). Conversely, we suppose that \( \{S_n : n \in \mathbb{N}_0\} \) has stationary, independent increments. Then, \( \{X_n : n \in \mathbb{N}\} \) is a sequence of iid random variables by looking at \( X_n = S_n - S_{n-1} \).

Lemma 1.14. If a simple point process \( \{N(t), t \geq 0\} \) has stationary and independent increments then the sequence of inter-arrival times \( \{X_n : n \in \mathbb{N}\} \) are iid random variables.

Proof. First, we notice that
\[
\{X_n > y\} = \{N(S_{n-1}) \leq N(S_{n-1} + y) < N(S_n)\} = \{N(S_{n-1} + y) - N(S_{n-1}) = 0\}.
\]

To show that each inter-arrival time is identically distributed, we utilize the stationary increment property of the counting process \( N(t) \), to observe
\[
\Pr\{S_n - S_{n-1} > x\} = \int_0^\infty \Pr\{N(x) = 0\}dF_{n-1}(t) = \Pr\{N(x) = 0\} = \Pr\{X_1 > x\}.
\]

To show that inter-arrival times are independent, it suffices to show that \( X_n \) is independent of \( S_{n-1} \). Using the independent increments property, we see that
\[
\Pr\{S_{n-1} \leq x, X_n > y\} = \int_0^x \Pr\{N(y + t) - N(t) = 0 | S_{n-1} = t\}dF_{n-1}(t)
= \int_0^x \Pr\{N(y + t) - N(t) = 0 | N(t) = n-1, N(s) < n-1, s < t\}dF_{n-1}(t)
= \Pr\{X_n > y\}F_{n-1}(x).
\]

2 Poisson process

Lemma 2.1. A unique non-negative right continuous function \( f : \mathbb{R} \to \mathbb{R} \) satisfying equation
\[
f(t + s) = f(t)f(s), \text{ for all } t, s \in \mathbb{R}
\]
is \( f(t) = e^{\theta t} \), where \( \theta = \log f(1) \).

Omit. Clearly, we have \( f(0) = f^2(0) \). Since \( f \) is non-negative, it means \( f(0) = 1 \). By definition of \( \theta \) and induction for \( m, n \in \mathbb{Z}^+ \), we see that
\[
f(m) = f(1)^m = e^{\theta m}, \quad e^\theta = f(1) = f(1/n)^n.
\]

Let \( q \in \mathbb{Q} \), then it can be written as \( m/n, n \neq 0 \) for some \( m, n \in \mathbb{Z}^+ \). Hence, it is clear that for all \( q \in \mathbb{Q}^+ \), we have \( f(q) = e^{\theta q} \), either unity or zero. Note, that \( f \) is a right continuous function and is non-negative. Now, we can show that \( f \) is exponential for any real positive \( t \) by taking a sequence of rational numbers \( \{t_n\} \) decreasing to \( t \). From right continuity of \( g \), we obtain
\[
g(t) = \lim_{t_n \downarrow t} g(t_n) = \lim_{t_n \downarrow t} e^{\beta t_n} = e^{\beta t}.
\]
Definition 2.2. A random variable $X$ with continuous support on $\mathbb{R}_+$, is called memoryless if for all $t, s \in \mathbb{R}_+$, we have

$$\Pr\{X > s\} = \Pr\{X > t + s | X > t\}.$$ 

Proposition 2.3. The unique memoryless distribution function with continuous support on $\mathbb{R}_+$ is the exponential distribution.

Proof. Let $X$ be a random variable with a distribution function $F : \mathbb{R}_+ \rightarrow [0, 1]$ with the memoryless property. Let $g(t) \equiv 1 - F(t)$. It follows from the memoryless property of $F$, that

$$g(t + s) = g(t)g(s).$$

Since $g(x) = \Pr\{X > x\}$ is non-increasing in $x \in \mathbb{R}_+$, we have $g(x) = e^{\theta x}$, where $\theta < 0$. □

Definition 2.4. A simple point process $\{N(t), t \geq 0\}$ is called a Poisson process with a finite positive rate $\lambda$, if inter-arrival times $\{X_n : n \in \mathbb{N}\}$ are iid random variables with an exponential distribution of rate $\lambda$. That is, it has a distribution function $F$, such that

$$F(x) = \Pr\{X \leq x\} = \begin{cases} 1 - e^{-\lambda x}, & x \geq 0 \\ 0, & \text{else} \end{cases}.$$ 

![Figure 2: Stationary independent increment property of Poisson process.](image)

Proposition 2.5. A Poisson process $\{N(t), t \geq 0\}$ is simple point process with stationary independent increments.

Proof. It is clear that Poisson process is a simple point process. To show that $N(t)$ has stationary independent increment property, it suffices to show that $N_t - N(s) \perp N(s)$ and $N(t) - N(s) \sim N(t - s)$. This follows from the fact that we can use induction to show stationary independent increment property for for any finite disjoint time-intervals.
Let arrival time-instants \( \{S_n : n \in \mathbb{N}_0\} \) and inter-arrival times \( \{X_n : n \in \mathbb{N}\} \) be defined as before. Given any time \( s \), we can define the following variables

\[
X'_{N(s)+1} = s - S_{N(s)}, \quad X''_{N(s)+1} = S_{N(s)+1} - s.
\]

It is clear that \( s \) partitions \( X'_{N(s)+1} \) in two parts such that \( X_{N(s)+1} = X'_{N(s)+1} + X''_{N(s)+1} \) as seen in Figure 2 for the case when \( N(s) = n \). We look at joint distribution of \( X'_{N(s)+1}, X''_{N(s)+1} \) and notice that

\[
\{X'_{N(s)+1} > x, X''_{N(s)+1} > y\} = \bigcup_{n \in \mathbb{N}_0} \{S_n < s - x, S_{n+1} > s + y, N(s) = n\}
\]

\[
= \bigcup_{n \in \mathbb{N}_0} \{S_n < s - x, S_{n+1} > s + y\}.
\]

From the fact that inter-arrival times are i.i.d exponentially distributed with rate \( \lambda \), we conclude that

\[
\Pr\{X'_{N(s)} > x, X''_{N(s)+1} > y\} = \sum_{n \in \mathbb{N}_0} \int_{u=0}^{s-x} \Pr\{X_{n+1} > s + y + u\}dF_n(u),
\]

\[
= \int_{u=0}^{s-x} (1 - F_1(s + y + u)) \sum_{n \in \mathbb{N}_0} dF_n(u) = \int_{u=0}^{s-x} e^{-\lambda(s+y+u)}\lambda du,
\]

\[
= (1 - F_1(y))(F_1(s) - F_1(2s - x)).
\]

Therefore, \( X''_{N(s)+1} \) is independent of \( X'_{N(s)+1} \) and has same distribution as \( X_{n+1} \). The memoryless property of exponential distribution is crucially used. Further, we see that independent increment holds only if inter-arrival time is exponential. Therefore,

\[
\{N(s) = n\} \iff \{S_n = s + X'_{n+1}\},
\]

\[
\{N(t) - N(s) \geq m\} \iff \{X''_{n+1} + \sum_{i=n+2}^{n+m} X_i \leq t - s\}.
\]

Since, \( \{X_i : i \geq n+2\} \cup \{X''_{n+1}\} \) are independent of \( \{X_i : i \leq n\} \cup X'_{n+1} \), we have \( N(t) - N(s) \perp N(s) \). Further, since \( X''_{n+1} \) has same distribution as \( X_{n+1} \), we get \( N(t) - N(s) \sim N(t - s) \). By induction we can extend this result to \( (N(t_n) - N(t_{n-1}), \ldots, N(s)) \).

**Theorem 2.6 (Characterization 1).** A simple stationary independent increment process is a Poisson process with parameter \( \lambda \) when

\[
\lim_{t \to 0} \frac{\Pr\{N(t) = 1\}}{t} = \lambda, \quad \lim_{t \to 0} \frac{\Pr\{N(t) \geq 2\}}{t} = 0.
\]

**Proof.** It suffices to show that first inter-arrival times \( X_1 \) is exponentially distributed with parameter \( \lambda \). Notice that

\[
P_0(t + s) = \Pr\{N(t + s) = 0, N(t) = 0\} = P_0(t)P_0(s).
\]

Using the conditions in the theorem, the result follows. \( \square \)
2.1 Distribution functions

**Lemma 2.7.** Moment generating function of arrival times $S_n$ is

$$
E[e^{\theta S_n}] = \begin{cases} 
\frac{\lambda^n}{(\lambda - \theta)^n}, & \theta < \lambda \\
\infty, & \theta \geq \lambda.
\end{cases}
$$

Distribution function of $S_n$ is given by

Proof. Since $S_n = \sum_{k=1}^{n} X_k$, where $X_k$ are iid, the moment generating function $E[e^{\theta S_n}]$ of $S_n$ is

$$
E[e^{\theta S_n}] = (E[e^{\theta X_1}])^n.
$$

Since each $X_k$ is iid exponential with rate $\lambda$, it is easy to see that moment generating function of inter-arrival time $X_1$ is

$$
E[e^{\theta X_1}] = \begin{cases} 
\frac{\lambda^n}{(\lambda - \theta)^n}, & \theta < \lambda \\
\infty, & \theta \geq \lambda.
\end{cases}
$$

\[ \square \]

**Theorem 2.8.** Density function of $S_n$ is Gamma distributed with parameters $n$ and $\lambda$. That is,

$$
f_n(s) = \frac{\lambda^n (\lambda s)^{n-1}}{(n-1)!} e^{-\lambda s}.
$$

Proof. Notice that $X_i$’s are iid and $S_1 = X_1$. In addition, we know that $S_n = X_n + S_{n-1}$. Since, $X_n$ is independent of $S_{n-1}$, we know that distribution of $S_n$ would be convolution of distribution of $S_{n-1}$ and $X_1$. Since $X_n$ and $S_1$ have identical distribution, we have $f_n = f_{n-1} * f_1$. The result follows from straightforward induction. \[ \square \]

**Theorem 2.9.** For each $t > 0$, the distribution of Poisson process $N(t)$ with parameter $\lambda$ is given by

$$
\Pr\{N(t) = n\} = e^{-\lambda t} \frac{\lambda^n t^n}{n!}.
$$

Further, $E[N(t)] = \lambda t$, explaining the rate parameter $\lambda$ for Poisson process.

Proof. Result follows from density of $S_n$ and recognizing that

$$
P_n(t) = F_n(t) - F_{n+1}(t).
$$

\[ \square \]

**Corollary 2.10.** Distribution of arrival times $S_n$ is

$$
F_n(t) = \sum_{j \geq n} P_j(t), \quad \sum_{n \in \mathbb{N}} F_n(t) = E[N(t)].
$$
Proof. First result follows from the telescopic sum and the second from the following observation.

$$\sum_{n \in \mathbb{N}} F_n(t) = \mathbb{E} \sum_{n \in \mathbb{N}} 1\{N(t) \geq n\} = \sum_{n \in \mathbb{N}} \Pr\{N(t) \geq n\} = \mathbb{E}N(t).$$

\[ \square \]

Remark 2.11. A Poisson process is not a stationary process. That is, the finite dimensional distributions are not shift invariant.

Lemma 2.12. For any finite time $t > 0$, a Poisson process is finite almost surely.

Proof. By strong law of large numbers, we have

$$\lim_{n \to \infty} \frac{S_n}{n} = \mathbb{E}[X_1] = \frac{1}{\lambda} \text{ a.s.}$$

Fix $t > 0$ and let $M = \{\omega \in \Omega : N(t)(\omega) = \infty\}$ be a subset of the sample space. Let $\omega \in M$, then $S_n(\omega) \leq t$ for all $n \in \mathbb{N}$. This implies $\limsup_n \frac{S_n}{n} = 0$ and $\omega \notin \{\lim_n \frac{S_n}{n} = \frac{1}{\lambda}\}$. Hence, the probability measure for set $M$ is zero.

\[ \square \]

Proposition 2.13 (Characterization 2). Let $\{I_i \subseteq \mathbb{R}_+ : 1 \leq i \leq k\}$ be a sequence of disjoint intervals. A stationary independent increment point process $\{N(t), t \geq 0\}$, such that $N(0) = 0$ is Poisson process iff

$$\Pr\left(\bigcap_{i=1}^{k} \{N(I_i) = n_i\}\right) = \prod_{i=1}^{k} \frac{(\lambda |I_i|)^{n_i}}{n_i!} e^{-\lambda |I_i|}.$$