1. Consider a last-in-first-out M/G/1 queueing system with pre-emptive service. Arrivals are Poisson at rate $\lambda$. The random service time of the $n$th arriving customer is denoted by $Y_n$, which is independent of the arrivals and iid with distribution function $G$. Each arriving customer starts getting immediate service, pre-empting any other customer that may be getting serviced by the server. The interrupted customer gets serviced again, after all the subsequent arrivals have been serviced. We assume that the queue has no customer at time 0, then the time instants when the system becomes empty are renewal instants. The time duration between the first arrival since the $i$th renewal and the $(i+1)$th renewal instant is defined as the $i$th busy period $B_i$ of this queue. Find the moment generation function $M_{B_1}(\theta) = E e^{-\theta B_1}$ of the busy period of this queue. [Hint: Consider the scenario that other customers can arrive while the first arriving customer is in service. Each arrival interrupts the service.]

2. Consider a population model that starts with $I$ individuals, and has Poisson immigration of new individuals with rate $\lambda$. Each individual can die in a small interval $(t, t+h)$ with probability $\mu h + o(h)$. Find the asymptotic distribution of population size. Find the transition matrix of the jump chain, and its stationary distribution.

3. Consider an M/M/1 queueing system with Poisson arrivals at rate $\lambda$. The random service time of the $n$th arriving customer is denoted by $Y_n$, which is independent of the arrivals and iid with exponential distribution of rate $\mu > \lambda$. Show that the departure process from the stationary M/M/1 queue is Poisson with rate $\lambda$. [Hint: Show that the departure process is stationary and the number of zero departures in a time duration of length $t$ is exponentially distributed with rate $\lambda$.]

4. Consider a queueing system in which there are $k$ servers and customers arrive according to a Poisson process with rate $\lambda$. If an arriving customers find all the $k$ servers busy, then it does not enter the system. The service times of server are assumed to be distributed according to some general distribution $G$. Assume that $G$ has a density $g$. Let $\lambda(t)$ denote the hazard rate function. That is the instantaneous probability density that a $t$ unit old service will end, is

$$\lambda(t) = \frac{g(t)}{G(t)}.$$ 

Let $x_i$ denote the service time of the customer being served by $i$th server. At any time $t$, we can order the service time of customers in service. That is, for $n \leq k$, we can order $x_1 \leq x_2 \leq \ldots x_n$, where $k-n$ servers are idle. Let $X(t) = x = (x_1, x_2, \ldots, x_n)$ be a stochastic process indexed by time $t \geq 0$. Show that the process $X(t)$ is a Markov process. Write the allowed transitions for this Markov process and the associated transition rates. In addition, show that the reversed process is also a $k$ server loss system with service distribution $G$ in which arrivals occur according to a Poisson process with rate $\lambda$. For the reversed process, the state at any time represents the ordered residual service times of customers in service currently. Find the equilibrium distribution $\pi(x)$, and the distribution of $n$ customers in the system.