1. We consider a machine, that has a random life in days, distributed with probability mass function $P = \{P_i : i \in \mathbb{N}\}$. We assume that a failed machine is replaced by a new one that has an independent random life, distributed with the identical mass function $P$. We assume that the mean life of a new machine is finite. Let $X = \{X_n : n \in \mathbb{N}\}$ be the age of the machine in use on $n$th day. Show that $X$ is an aperiodic and irreducible Markov chain, and find the associated transition probability matrix and the equilibrium distribution in terms of mass function $P$.

2. Consider a discrete-time evolution of a population denoted by $X = \{X_n : n \in \mathbb{N}\}$. Life of each member of the population is iid geometric with probability of death $p$. The number of new members joining the population in each time-slot is an iid random variable with Poisson distribution of rate $\lambda$. Show that $X$ is an aperiodic and irreducible Markov chain, and find its stationary distribution.

3. Consider a set of $N$ distinct integers, and we want to find the number of trials to find the maximum of these integers, where we employ a random search scheme. In each trial, one randomly and uniformly picks one integer out of these $N$ integers. Find the mean and variance of the time to find the maximum value.

4. Let $\{X_n, n \in \mathbb{N}\}$ be an irreducible discrete time Markov chain with transition matrix $P$ such that $P^2 = P$. Show that
   
   (a) this Markov chain is aperiodic, and

   (b) transition probability is identical in every row, $P_{ij} = P_{jj}$, $\forall i, j$.

5. Consider a discrete Bernoulli process $\{Y_n \in \{0, 1\} : n \in \mathbb{N}\}$, where each $Y_n$ is iid Bernoulli with success probability $p = \Pr\{Y_n = 1\}$. Find the mean time between two consecutive observations of string 00.