1. The life of a car, $X$, is a random variable with distribution $F$. An individual has a policy of trading in his car either when it fails or reaches the age of $A$. Let $R(A)$ denote the resale value of an $A$-year-old car. There is no resale value for a failed car. Let $C_1$ denote the cost of a new car and suppose that an additional cost $C_2$ is incurred whenever the car fails.

(a) Say that a cycle begins each time a new car is purchased. Compute the long-run average cost per unit time.

(b) Say that a cycle begins each time a car in use fails. Compute the long-run average cost per unit time.

**Solution:** In both cases, you are expected to compute the ratio of the expected cost incurred in a cycle to the expected time of a cycle. The answer should be the same in both parts.

(a)

Expected cost per cycle, $E[C] = C_1 + C_2 F(A) - R(A) F^c(A)$

Expected length of cycle, $E[T] = E[T|X > A] P(X > A) + E[T|X \leq A] P(X \leq A)$

$$= E[A|X > A] P(X > A) + E[X|X \leq A] P(X \leq A)$$

$$= A F^c(A) + \int_0^A x dF_X(x)$$

By renewal reward theorem,

$$\lim_{t \to \infty} \frac{E[C(t)]}{t} = \frac{E[C]}{E[T]} = \frac{C_1 + C_2 F(A) - R(A) F^c(A)}{A F^c(A) + \int_0^A x dF_X(x)}$$

(b) The chance that a car fails is $F(A)$, so the number, $N$, of cars bought between failures has the geometric distribution with parameter $p = F(A)$. We have,

Expected cost per cycle, $E[C] = E[NC_1 - (N - 1)R(A) + C_2]$

$$= \frac{C_1 - R(A)}{F(A)} + R(A) + C_2$$

Expected length of cycle, $E[T] = E[(N - 1)A] + E[X|X \leq A]$

$$= A F^c(A) + \int_0^A x dF_X(x)$$

Again by renewal reward theorem,

$$\lim_{t \to \infty} \frac{E[C(t)]}{t} = \frac{E[C]}{E[T]} = \frac{\frac{C_1 - R(A)}{F(A)} + R(A) + C_2}{\frac{A F^c(A)}{F(A)} + \int_0^A x dF_X(x)} = \frac{C_1 + C_2 F(A) - R(A) F^c(A)}{A F^c(A) + \int_0^A x dF_X(x)}$$
2. Customers arrive at a bus stop according to a Poisson process of rate $\lambda$. Independently, buses arrive according to a renewal process with the inter-renewal interval c.d.f $F_X(x)$. At the epoch of a bus arrival, all waiting passengers enter the bus and the bus leaves immediately. Let $R(t)$ be the number of customers waiting at time $t$.

(a) Given that the first bus arrives at time $X_1 = x$, find the expected number of customers picked up; then find $\mathbb{E} \left[ \int_0^x R(t) dt \right]$, again given the first bus arrival at $X_1 = x$.

(b) Assuming $F_X$ is a non-arithmetic distribution, find $\lim_{t \to \infty} \mathbb{E}[R(t)]$.

Hint: For a non-arithmetic distribution, $\lim_{t \to \infty} \mathbb{E}[R(t)] = \lim_{t \to \infty} \frac{1}{t} \int_0^t R(\tau) d\tau$, w.p. 1.

(c) Find the fraction of time that there are no customers at the bus stop.

Solution:

(a) The expected number of customers picked up given $X_1 = x$, is the expected number of arrivals in the Poisson process by $x$, i.e., $\lambda x$. For $t < x$, $R(t)$ is the number of customers that have arrived by time $t$. $R(t)$, for $t < X_1$, is independent of $X_1$, so

$$\mathbb{E} \left[ \int_0^x R(t) dt | X_1 = x \right] = \int_0^x \mathbb{E}[R(t)|X_1 = x] dt = \int_0^x \mathbb{E}[R(t)] dt$$

$$= \int_0^x \lambda t dt = \frac{\lambda x^2}{2}$$

(b) From above,

$$\mathbb{E} \left[ \int_0^{X_1} R(t) dt \right] = \mathbb{E} \left[ \mathbb{E} \left( \int_0^{X_1} R(t) dt | X_1 = x \right) \right] = \frac{\lambda \mathbb{E}[X_1^2]}{2}$$

and for a non-arithmetic distribution, w.p. 1,

$$\lim_{t \to \infty} \mathbb{E}[R(t)] = \lim_{t \to \infty} \frac{1}{t} \int_0^t R(\tau) d\tau = \frac{1}{\mathbb{E}[X_1]} \mathbb{E} \left[ \int_0^{X_1} R(t) dt \right] = \frac{\lambda \mathbb{E}[X_1^2]}{2\mathbb{E}[X_1]}$$

(c) There are no customers at the bus stop at the beginning of each renewal period. Let $U_n$ be the interval from the beginning of the $n^{th}$ renewal period until the first customer arrival. It is possible that no customer will arrive before the next bus arrives, so the interval within the $n^{th}$ inter-renewal period when there is no customer waiting is $\min(U_n, X_n)$. Consider a reward function $R(t)$ which is equal to 1 when no customer is in the system and 0 otherwise. The accumulated reward $R_n$ within the $n^{th}$ inter-renewal period is then $R_n = \min(U_n, X_n)$.

Thus, using the independence of $U_n$ and $X_n$

$$\mathbb{E}[R_n] = \int_0^\infty P\{\min(U_n, X_n) > t\} dt = \int_0^\infty P\{U_n > t\} P\{X_n > t\} dt = \int_0^\infty e^{-\lambda t} P\{X_n > t\} dt$$

Therefore, the limiting time-average fraction of time when no customers are waiting is given by

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t R(\tau) d\tau = \frac{\mathbb{E}[R_1]}{\mathbb{E}[X_1]} = \frac{1}{\mathbb{E}[X_1]} \int_0^\infty e^{-\lambda t} P\{X_1 > t\} dt$$
3. Let \((X_n, n \geq 0)\) be a homogeneous Markov chain with state space \(S\) and transition probabilities \(P_{ij}, (i, j) \in S\). Let \(T_{ij}\) denote the number of transitions needed to hit state \(j\) for the first time, given that the Markov chain starts off at state \(i\). If two states \(i\) and \(j\) communicate, show using Renewal reward theorem,

(a) \(\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} P_{ij}^k = \frac{1}{E[T_{jj}]}\).

(b) If \(j\) is aperiodic, then \(\lim_{n \to \infty} P_{ij}^n = \frac{1}{E[T_{jj}]}\).

(c) If \(j\) has period \(d\), then \(\lim_{n \to \infty} P_{ij}^{nd} = dE[T_{jj}]\).

Solution:

(a) Let \(N(t)\) be a function which is equal to 1 when the Markov chain is in state \(j\) and 0 otherwise. Then \(N(t)\) is a delayed renewal process with expected inter-arrival time \(Y(1) \sim T_{ij}\) and \(Y(k) \sim T_{jj}\) for \(k \geq 2\). Then by Renewal reward theorem,

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} P_{ij}^k = \lim_{n \to \infty} \frac{1}{n} E \left[ \sum_{k=1}^{n} 1\{X(k)=j|X(0)=i\} \right] = \lim_{n \to \infty} \frac{1}{n} E[N(n)] = \frac{1}{E[T_{jj}]}.
\]

(b) \(\lim_{n \to \infty} P_{ij}^n = \lim_{n \to \infty} E \left[ 1\{X(n)=j|X(0)=i\} \right] = \lim_{n \to \infty} E \left[ \sum_{k=1}^{n} 1\{X(k)=j|X(0)=i\} - \sum_{k=1}^{n-1} 1\{X(k)=j|X(0)=i\} \right] = \lim_{n \to \infty} E[N(n)] - E[N(n-1)] = \frac{1}{E[T_{jj}]}\) \{By Blackwell’s renewal theorem\}

(c) \(\lim_{n \to \infty} P_{ij}^{nd} = \lim_{n \to \infty} E \left[ 1\{Renewal\ at\ time\ nd\} \right] = \lim_{n \to \infty} E \left[ Number\ of\ renewals\ at\ time\ nd\right] = \frac{d}{E[T_{jj}]}\) \{By Blackwell’s renewal theorem\}

4. Consider an undirected graph with finite number of vertices having a positive number \(w_{ij}\) associated with each edge \((i, j) \in E\) and suppose that a particle moves from vertex \(i\) to vertex \(j\) with transition probability \(P_{ij} = \frac{w_{ij}}{\sum_k w_{ik}}\), where \(w_{ij} = 0\) if \((i, j)\) is not an edge of the graph. The Markov chain describing the sequence of vertices visited by the particle is called a random walk on an edge-weighted graph.

(a) Show that if the graph is connected, then the Markov chain is irreducible.

(b) Assume the graph is connected, find the stationary probability distribution of the random walk on the corresponding edge-weighted graph.
Solution:

(a) If the graph is connected then between any two vertices $i$ and $j$, $i, j \in \mathcal{V}$, there exists a path $i \rightarrow k_1 \rightarrow k_2 \rightarrow \ldots \rightarrow k_n \rightarrow j$, which implies the associated weights $w_{i,k_1}, w_{k_1,k_2}, \ldots, w_{k_{n-1},k_n}, w_{k_n,j}$ are non-zero. Hence, $P_{i,k_1}P_{k_1,k_2}\ldots P_{k_{n-1},k_n}P_{k_n,j} > 0$. Hence, $i$ and $j$ communicate and the Markov chain is irreducible. Similar argument can be used to show irreducibility implies connectedness.

(b) We can solve for $\pi = \pi P$. We have, $\sum_{i \in \mathcal{V}} \pi(i)P(i,j) = \sum_{i \in \mathcal{V}} \pi(i)\frac{w_{ij}}{\sum_{k \in \mathcal{V}} w_{ik}}$, which motivates solutions of the form, $\pi(k) = C \sum_{l \in \mathcal{V}} w_{lk}$. Then

$$(\pi P)(j) = \sum_{i \in \mathcal{V}} \pi(i) \frac{w_{ij}}{\sum_{k \in \mathcal{V}} w_{ik}} = \sum_{i \in \mathcal{V}} C w_{ij} = C \sum_{i \in \mathcal{V}} w_{ji} \{ \text{Since, } w_{ij} = w_{ji} \} = \pi(j)$$

To find $C$ we use, $\sum_{k \in \mathcal{V}} \pi(k) = 1$, which gives $C = \frac{1}{\sum_{k \in \mathcal{V}} w_{jk}}$. Hence, $\pi(i) = \frac{\sum_{l \in \mathcal{V}} w_{il}}{\sum_{k \in \mathcal{V}} w_{lk}}$

Alternatively, we can use the notion of reversibility, i.e., if there exists $\pi$ that satisfies the detailed balance equation (DBE), $\pi(i)P(i,j) = \pi(j)P(j,i)$ for all $i, j \in \mathcal{V}$, then the $\pi$ is the stationary probability vector for the graph and the associated Markov chain is reversible. From the DBE,

$$\pi(i) \frac{w_{ij}}{\sum_{k \in \mathcal{V}} w_{ik}} = \pi(j) \frac{w_{ji}}{\sum_{l \in \mathcal{V}} w_{lj}}$$

$$\pi(i) \sum_{k \in \mathcal{V}} w_{jk} = \pi(j) \sum_{k \in \mathcal{V}} w_{ik} \{ \text{Since, } w_{ij} = w_{ji} \}$$

Summation over all $j \in \mathcal{V}$, then gives

$$\pi(i) \sum_{j, l \in \mathcal{V}} w_{jl} = \left( \sum_{j \in \mathcal{V}} \pi(j) \right) \sum_{k \in \mathcal{V}} w_{ik}$$

$$\therefore \pi(i) = \frac{\sum_{k \in \mathcal{V}} w_{ik}}{\sum_{j \in \mathcal{V}} w_{ji}}$$

5. Consider a gambler who at each play of the game has probability $p$ of winning 1 unit and probability $q = 1 - p$ of losing 1 unit. The gambler quits playing either when he has zero or $N$ units. Assume the successive plays of the game are independent.

(a) Let $X_n$ denote the player’s fortune at time $n$. Argue that $\{X_n, n \in 0, 1, 2, \ldots \}$ is a Markov chain, find its transition probabilities and identify the set of recurrent and transient states.

(b) Let $f_i$ denote the probability that, starting with $i, 0 < i < N$, the gambler’s fortune will eventu-
ally reach N. Show that \( f_i = pf_{i+1} + qf_{i-1} \) and

\[
f_i = \begin{cases} 
1 - (q/p)^i & \text{if } p \neq q \\
\frac{i}{N} & \text{if } p = q
\end{cases}
\]

*Hint: Condition on the outcome of the initial play of the game. To solve for \( f_i \) consider solutions of the form \( f_i = A + B\lambda^i \).*

(c) What is the expected number of plays that the gambler, starting with \( i \) units, makes before reaching either zero or \( N \).

*Hint: Apply Wald’s Lemma.*

**Solution:**

\[ 1 \rightarrow 0 \rightarrow 1 \rightarrow 2 \ldots \rightarrow i-1 \rightarrow i \rightarrow i+1 \ldots \rightarrow N-1 \rightarrow N \rightarrow 1 \]

(a) Assume, \( X_0 = i \), for some \( i \geq 1 \). Clearly, 0 and \( N \) are the only recurrent states, all other states are transient. The transition probabilities are as shown in the state transition diagram.

(b) Let \( Y_k \) denote the outcome of the \( k^{th} \) play. Then, \( P[Y_k = 1] = p = 1 - P[Y_k = -1] \). Let \( T \) be the play at which the gambler stop playing, i.e., \( T = \min \{ n \in \mathcal{N} : \sum_{k=1}^{n} Y_k = 0 \text{ or } N \} \).

Then

\[
f_i = P[X_T = N | X_0 = i] = P[X_T = N, Y_k = 1 | X_0 = i] + P[X_T = N, Y_k = -1 | X_0 = i]
\]

\[
= P[Y_k = 1 | X_0 = i] P[X_T = N | X_0 = i, Y_k = 1] + P[Y_k = -1 | X_0 = i] P[X_T = N | X_0 = i, Y_k = -1]
\]

\[
= P[Y_k = 1 | X_0 = i] P[X_T = N | X_0 = i, X_1 = i+1] + P[Y_k = -1 | X_0 = i] P[X_T = N | X_0 = i, X_1 = i-1]
\]

\[
= P[Y_k = 1] P[X_T = N | X_1 = i+1] + P[Y_k = -1] P[X_T = N | X_1 = i-1] = pf_{i+1} + (1-p)f_{i-1}
\]

To solve for \( f_i \) when \( p \neq q \), we assume, \( f_i = A + B\lambda^i \). From the boundary conditions \( f_0 = 0 \) and \( f_N = 1 \), we get \( B = -A \) and \( A = \frac{1}{1-\lambda^N} \). Thus, \( f_i = A \left[ 1 - \lambda^i \right] \). Then from the difference equation,

\[
(p+q)f_i = pf_{i+1} + (1-p)f_{i-1}
\]

\[
p[f_i - f_{i+1}] = (1-p)[f_{i-1} - f_i]
\]

\[
pA\lambda^i[1-\lambda] = (1-p)A\lambda^{i-1}[1-\lambda]
\]

\[
\lambda = \frac{(1-p)}{p}
\]

Thus, \( f_i = \frac{1 - (\frac{q}{p})^i}{1 - (\frac{q}{p})} \). Similarly, try a solution of the form \( f_i = A + B\lambda^k + Ck \) to solve for the case \( p = q \).
(c) We have $X_T = i + N$ with probability $f_i$ and $X_T = 0$ with probability $1 - f_i$. Note that $E[X_T] = E\left[\sum_{k=1}^{T} Y_k \right]$. Applying Wald’s Lemma

$$E[X_T] = E[T] E[Y_k]$$

$$= E[T] (p - q)$$

$$\therefore E[T] = \frac{1}{p - q} E[X_T]$$

$$= \frac{1}{p - q} [f_i (i + N)] = \frac{i + N}{p - q} \left[ \frac{1 - \left( \frac{q}{p} \right)^i}{1 - \left( \frac{q}{p} \right)^N} \right]$$

Figure 1: Transition probabilities of the Star graph

Using the results from the previous two questions, solve the following:

6. Consider a star graph consisting of $r$ rays, with each ray consisting of $n$ vertices. Let leaf $i$ denote the leaf on ray $i$. Assume that a particle moves along the vertices of the graph in the following manner. Whenever it is at the central vertex 0, it is then equally likely to move to any of its neighbors. Whenever it is on an internal (non-leaf) vertex of a ray, then it moves towards the leaf of that ray with probability $p$ and towards 0 with probability $1 - p$. Whenever it is at a leaf, it moves to its
neighbor vertex with probability 1. Say, a new cycle begins whenever the particle returns to vertex zero. Let $X_j$ be the number of transitions in the $j$-th cycle, $j \geq 1$. Fix $i$ and let $N(i)$ denote the number of cycles that it takes for the particle to visit leaf $i$ and then return to 0.

(a) Construct an undirected edge-weighted graph corresponding to the star graph. 

Hint: Note that, $w_{ij} = f(i)P_{ij}$. Take $f(i) = c_i$ and use the condition $w(i, j) = w(j, i)$ for all $(i, j) \in E$.

(b) Find the expected number of steps between returns to central vertex 0.

(c) Find $E[N(i)]$.

Hint: Each cycle will independently reach leaf $i$ with probability $\frac{1}{r}$ times the probability (Gambler’s ruin) that a particle on the first vertex of ray $i$ will reach the leaf of that ray (that is, increase by $n - 1$) before returning to zero.

(d) Starting at vertex 0, find the expected number of transitions that it takes to visit all the vertices and then return to 0.

---

**Solution:**

(a) Note that the non-zero probabilities are $P_{0,\text{ray } i}$, $P_{i, i+1}, 1 \leq i \leq n - 1$, $P_{i, j}, 1 \leq j \leq n$. Consider weights of the form, $w_{ij} = f(i)P_{ij}$. To find $f(k)$, $0 \leq k \leq n$, we use the conditions $w_{ij} = w_{ji}$ for all $(i, j) \in E$, we obtain the following equations,

$$
\begin{align*}
&w_{0, \text{ray } i} = w_{\text{ray } i, 0} \\
&w_{i, i+1} = w_{i+1, i} \\
&w_{j-1, j} = w_{j, j-1} \\
&w_{n-1, n} = w_{n, n-1}
\end{align*}
$$

from which we obtain,

$$
\frac{f(0)}{f(1)} = qr, \quad \frac{f(i)}{f(i+1)} = q, \quad \frac{f(n-1)}{f(n)} = \frac{1}{p}
$$

Taking product of above equations gives, $f(k) = \frac{1}{qr} \left( \frac{p}{q} \right)^{k-1} f(0)$ for $1 \leq k \leq n - 1$ and $f(n) = \frac{1}{qr} \left( \frac{p}{q} \right)^{n-2} f(0)$. Unlike probabilities, weights need not be normalized, hence $f(0)$ can be chosen arbitrary. Choosing $f(0) = qr$, then gives

$$
\begin{align*}
&f(k) = \left( \frac{p}{q} \right)^{k-1}; \quad \text{for } 1 \leq k \leq n - 1 \\
&f(n) = \frac{q}{p^2} \left( \frac{p}{q} \right)^{n-1}
\end{align*}
$$
(b) For the undirected weighted edge graph,

\[ \pi(0) = \frac{\sum_{l \in \text{neighbors}(0)} w_{0,l}}{\sum_{k,l} w_{k,l}} \]

\[ = r f(0) P_{0,\text{ray 1}} \]

\[ = r \left[ \frac{f(0)}{r} + \frac{n-1}{1} \sum_{k=1}^{n-1} f(k) P_{,k+1} + \frac{n-1}{1} \sum_{l=1}^{n-1} f(l) P_{,l-1} + f(n) P_{,n,n-1} \right] \]

\[ = \frac{f(0)}{r} \left[ \frac{1 - \frac{r}{q}}{2} \right] \]

Note that for a Markov chain corresponding to successive states visited, returns to state 0 forms renewal instants. Hence, by renewal reward theorem, expected number of steps between returns to central vertex 0, \( \mu_{00} = \frac{1}{\pi(0)} = \frac{2[1-(\frac{q}{r})^n]}{(1-\frac{r}{q})} \).

(c) For any given node \( i \), the probability that a cycle enters ray \( i \) is \( r \). Also, the probability that the Markov chain on entering ray \( i \) will visit leaf \( i \) before returning to vertex zero is the same as the probability of earning \( n-1 \) units starting with one unit in a Gambler’s ruin problem, i.e., \( f_1 = \frac{(1-\frac{q}{r})}{1-\frac{p}{q}} \). Thus, \( N(i) \) is geometrically distributed with probability of success \( \frac{(1-\frac{q}{r})}{1-\frac{p}{q}} \). Thus, \( E[N(i)] = \frac{r[1-(\frac{q}{r})^n]}{(1-\frac{p}{q})} \).

It is instructive to also calculate the expected number of transitions required to visit leaf \( i \) and return to 0. The number of transitions required to visit leaf \( i \) is equal to \( N(i) \sum_{k=1}^{N(i)} X_k \). Since,

\[ E[N(i)] \mu_{00} = 2r \left( \frac{q}{p} \right) \left[ \frac{1-(\frac{q}{r})^n}{(1-\frac{p}{q})} \right] \]

(d) For the solution, we will use results from the Coupon collecting problem.

**Coupon Collecting problem:**

A company issues \( n \) different types of coupons. A collector desires a complete set. We suppose each coupon he acquires is equally likely to be each of the \( n \) types. How many coupons must he obtain on an average so that his collection contains all \( n \) types?

Let \( Z_t \) denote the number of different types represented among the collectors first \( t \) coupons. Clearly \( Z_0 = 0 \). When the collector has coupons of \( k \) different types, there are \( n-k \) types missing. Of the \( n \) possibilities for his next coupon, only \( n-k \) will expand his collection. Thus,

\[ P\{Z_{t+1} = k+1 | Z_t = k\} = \frac{n-k}{n} \]

\[ P\{Z_{t+1} = k | Z_t = k\} = \frac{k}{n} \]
Let $\tau$ be the (random) number of coupons collected when the set first contains every type. The expectation $\mathbb{E}(\tau)$ can be computed by writing $\tau$ as a sum of geometric random variables. Let $\tau_k$ be the total number of coupons accumulated when the collection first contains $k$ distinct coupons. Then, $\tau = \tau_n = \tau_1 + (\tau_2 - \tau_1) + \cdots + (\tau_n - \tau_{n-1})$.

Note that, $\tau_k - \tau_{k-1}$ is a geometric random variable with success probability $\frac{n-k+1}{n}$ because after collecting $k-1$ coupons, there are $n-(k-1)$ types missing from the collection. Thus each subsequent coupon drawn has the same probability $\frac{n-k+1}{n}$ of being a type not already collected, until a new type is finally drawn. Thus, $\mathbb{E}(\tau_k - \tau_{k-1}) = \frac{n}{n-k+1}$.

\[
\mathbb{E}[\tau] = \sum_{k=1}^{n} \mathbb{E}[\tau_k - \tau_{k-1}] = n \sum_{k=1}^{n} \frac{1}{n-k+1} = n \sum_{k=1}^{n} \frac{1}{k}
\]

In the Star graph problem, visiting a distinct leaf node corresponds to collecting a new type of coupon. Let $N_k$ denote the number of cycles needed to visit $k$ distinct leafs, then

\[
T_k = \sum_{j=1}^{N_k} X_j
\]

Then,

\[
T = T_r = \sum_{k=1}^{r} T_k - T_{k-1} = \sum_{k=1}^{r} \sum_{j=N_k-1}^{N_k} X_j = \sum_{k=1}^{r} \sum_{j=1}^{N_k-N_{k-1}} X_j
\]

Here, $N_k - N_{k-1}$ is geometrically distributed with probability of success

\[
\left( \frac{r-k+1}{r} \right) \left( \frac{1 - \frac{q}{p}}{1 - \left( \frac{q}{p} \right)^n} \right)
\]

which is the product of the probability of entering a distinct ray and the probability of visiting the leaf of that ray. Hence, the expected time to visit all leaves and return to 0, $\mathbb{E}[T]$ is given by

\[
\mathbb{E}[T] = \mathbb{E} \left[ \sum_{k=1}^{r} \sum_{j=1}^{N_k-N_{k-1}} X_j \right] = \sum_{k=1}^{r} \mathbb{E} \left[ \sum_{j=1}^{N_k-N_{k-1}} X_j \right]
\]

\[
= \sum_{k=1}^{r} \mathbb{E} [N_k - N_{k-1}] \mathbb{E} [X_j] \quad \text{\{ Wald’s Lemma \}}
\]

\[
= \sum_{k=1}^{r} \left( \frac{r}{r-k+1} \right) \left( 1 - \left( \frac{q}{p} \right)^n \right) \mu_0
\]

\[
= r \left( \frac{1 - \left( \frac{q}{p} \right)^n}{1 - \frac{q}{p}} \right) \mu_0 \sum_{k=1}^{r} \frac{1}{r-k+1}
\]

\[
= 2r \left( \frac{q}{p} \right)^{n-1} \left( \frac{1 - \left( \frac{q}{p} \right)^n}{1 - \frac{q}{p}} \right) \left( \sum_{k=1}^{r} \frac{1}{k} \right)
\]