1. Consider a queue where the customers are arriving according to a Poisson process with rate $\lambda$. Compute the aggregate expected waiting time for all arriving customers by time $t$.

**Solution:** We denote the $n$th arrival instant of the counting process $N(t)$ by $S_n$. If $N(t) = n$, then the total expected waiting time is given by

$$
\mathbb{E} \left( \sum_{i=1}^{N(t)} (t - S_i) \right) = \mathbb{E} \left( \sum_{i=1}^{n} (t - S_i) \big| N(t) = n \right) \tag{1}
$$

Recall that given the number of arrivals in a particular time duration, the arrivals are order statistics of uniformly distributed random variables in that time interval. Let $U_1, U_2, \ldots, U_n$ be iid uniform on $[0, t]$, and let $U_{(1)}, U_{(2)}, \ldots, U_{(n)}$ be their order statistics. Then

$$
\mathbb{E} \left[ \sum_{i=1}^{n} (t - U_{(i)}) \big| N(t) = n \right] = nt - \mathbb{E} \left[ \sum_{i=1}^{n} U_{(i)} \big| N(t) = n \right] = \frac{nt}{2}.
$$

Substituting conditional expectation in equation (1), we obtain

$$
\mathbb{E} \left( \sum_{i=1}^{N(t)} (t - S_i) \right) = \mathbb{E}[N(t)] \frac{t}{2} = \frac{\lambda t^2}{2}.
$$

2. Let $\{Z(t) : t \geq 0\}$ be a compound Poisson process whose jumps occur at rate $\lambda$ and the iid jump sizes $\{X_i : i \in \mathbb{N}\}$ are of discrete sizes, taking values in a countable set $E \subset \mathbb{R}$. Compute the mean function $\mathbb{E}Z(t)$ and the moment generating function $\mathbb{E}e^{-\theta Z(t)}$. Compute the moment generating function when jump sizes are not necessarily discrete valued.

**Solution:** If $X_i$ iid with mean $\mu$, then we can find mean off $Z_t$ as

$$
\mathbb{E}[Z_t] = \mathbb{E}\left[\mathbb{E}[Z|N_t]\right] = \mathbb{E}\left[\sum_{i=1}^{N_t} X_i|N_t\right] = \mu \mathbb{E}N_t = \lambda \mu t.
$$

For an element $e \in E$, let $N_e(t)$ be the number of jumps of size $e$ in time $[0, t)$. Observe that $\{N_e(t), e \in E\}$ are independent Poisson with rate $\{\lambda_e : e \in E\}$ where $\lambda_e = \lambda \Pr\{X_i = e\}$. We can write $Z(t) = \sum_{e \in E} e N_e(t)$, to obtain

$$
\mathbb{E}\left[ e^{-\theta Z(t)} \right] = \mathbb{E}\left[ \prod_{e \in E} e^{-\theta e N_e(t)} \right] = \prod_{e \in E} \mathbb{E}\left[ e^{-\theta e N_e(t)} \right] \tag{2}
$$

However, we have

$$
\mathbb{E}\left[ e^{-\theta N(t)} \right] = \sum_{n=0}^{\infty} e^{-\theta n} e^{-\lambda t} \frac{(\lambda t)^n}{n!} = e^{\lambda t(1-e^{-\theta})}.
$$

Substituting this back in equation (3), we obtain

$$
\mathbb{E}\left[ e^{-\theta Z(t)} \right] = \prod_{e \in E} e^{-\lambda t(1-e^{-\theta e})} = \exp\left[ -t \sum_{e \in E} \lambda (1-e^{-\theta e}) \Pr\{X_i = e\} \right].
$$

If $X_i$ are iid with a distribution function $\varphi$, then we can write moment generating function of $Z(t)$ in terms of moment generating function $M_X(\theta) = \mathbb{E}[e^{\theta X}]$ of $X_1$ as

$$
\mathbb{E}\left[ e^{-\theta Z(t)} \right] = \mathbb{E}\left[ e^{-\theta \sum_{i=1}^{N(t)} X_i} \right] = \sum_{n=0}^{\infty} e^{\lambda t} \frac{(-\lambda)^n}{n!} M_X(\theta)^n = e^{-\lambda t(1-M_X(\theta))} = \exp\left[ -t \int_{0}^{\infty} (1-e^{-\theta u}) \lambda d\varphi(u) \right].
$$
3. Consider a Poisson arrival with rate $\lambda$, where each arrival can be tagged to be of type $i \in [k]$. Let $Y_i(s)$ be the indicator that an arrival at time $s$ can be classified to be of type $i$. For each $s$, we assume $Y_i(s)$ to be a random variable independent of everything else. We denote $p_i(s) = \mathbb{E}Y_i(s)$ such that $\sum_i p_i(s) = 1$. Let $N_i(t)$ denote the number of arrivals of type $i$ in time interval $[0,t]$. Show that \{ $N_i(t) : i \in [k]$ \} has a Poisson distribution of mean $\lambda \int_0^t p_i(s)ds$ for type $i$.

Solution: Let $N(t) = n$ and call $S_1, S_2, \ldots, S_n$ be the arrival instants. For $i \in [k]$, let $Y_i(s)$ be an indicator variable at time $s$ with $\mathbb{E}Y_i(s) = p_i(s)$. Clearly, we can write

$$N_i(t) = \sum_{n \in \mathbb{N}} 1\{S_n \leq t\}Y_i(S_n)$$

Therefore, denoting $\Pr\{S_n \leq t\}$ by $F_n(t)$ and the fact that $m(t) = \lambda t = \sum_{n \in \mathbb{N}} F_n(t)$, it follows that

$$\mathbb{E}N_i(t) = \sum_{n \in \mathbb{N}} \int_0^t dF_n(u)p_i(u) = \int_0^t \sum_{n \in \mathbb{N}} dF_n(u)p_i(u) = \int_0^t dm(u)p_i(u) = \lambda \int_0^t p_i(u)du.$$ 

The result follows since $N_i(t)$ has independent increments.

4. Let $S_n$ be the $n$th jump instant of a renewal process with iid inter-renewal time $X_n$ and renewal function $m$.

(a) Compute $\sum_{n \in \mathbb{N}} P\{S_n \leq t\}$.

(b) Compute $\mathbb{E}e^{-\theta S_n}$.

(c) Compute $\mathbb{E}\sum_{n \in \mathbb{N}} f(S_n)$ for any non-negative function on $\mathbb{R}_+$.

Solution:

(a) 

$$\sum_{n \in \mathbb{N}} P\{S_n \leq t\} = \sum_{n \in \mathbb{N}} P\{N(t) \geq n\} = \mathbb{E}[N(t)] = m(t).$$

(b) Since the inter-arrival times of a Poisson process are iid, it follows that for some $\lambda > 0$, we have

$$\mathbb{E}[e^{-\theta S_1}] = [e^{-\theta X}]^n = M_X(\theta)^n.$$

(c) Let $f$ be any non-negative function on $\mathbb{R}_+$. Then, using the density of $S_n$ and exchanging infinite summation and integration using Monotone convergence theorem, we get

$$\mathbb{E}\sum_{n \in \mathbb{N}} f(S_n) = \sum_{n \in \mathbb{N}} \int_{\mathbb{R}_+} f(t)dF_{S_n}(t) = \int_{\mathbb{R}_+} f(t)d(\sum_{n \in \mathbb{N}} F_{S_n}(t)) = \int_{\mathbb{R}_+} f(t)dm(t).$$