1. Let \( \{X_t, t \geq 0\} \) be a stochastic process such that
\[
X_t = A \cos(2\pi t + \Theta), \quad t \geq 0.
\]
If \( A \) and \( \Theta \) are independent random variables, and \( \Theta \) is uniformly distributed between \((-\pi, \pi]\).

(a) Compute the following functions \( m_X(t), R_X(t,s) \), and \( C_X(t,s) \).

(b) Compute the finite dimensional distribution of \( X_t \) when \( A \) is uniform between \([0, 1]\).

(c) Is the process stationary? In what sense?

**Solution:**

(a) From independence of \( A \) and \( \Theta \) and uniformity of \( \Theta \) is follows that
\[
m_X(t) = E(A \cos(2\pi t + \Theta)) = E(A)E(\cos(2\pi t + \Theta)) = 0.
\]

Since \( m_X(t) = 0 \) for all \( t \geq 0 \), it follows that \( R_X(t,s) = C_X(t,s) \). Further, we can compute from independence of \( A \) and \( \Theta \),
\[
R_X(t,s) = E(\frac{A^2}{2} \cos(2\pi (t-s) + \cos(2\pi (t+s) + 2\Theta)) = E(\frac{A^2}{2} \cos(2\pi (t-s))\cos(2\pi t + 2\Theta))
\]

(b) Let \( S \subset \mathbb{R}_+ \) be a finite subset such that \( S = \{t_1, t_2, \ldots, t_k\} \) has ordered elements. Let \( x = (x_1, x_2, \ldots, x_k) \) where each \( x_i \in (0, 1) \) for all \( i \in [k] \). Then, we can write
\[
F_S(x \leq x) = P\{A \cos(2\pi t_i + \Theta) \leq x_i, i \in [k]\} = \frac{1}{2\pi} I^1_{\int_0^{t_i}} \prod_{i=1}^{k} \frac{1}{\cos^{-1}(x_i/a) - 2\pi i} d\theta.
\]

(c) Process is not stationary, since the finite dimensional distributions are not shift invariant. However, it is wide-sense stationary, since mean and second moment are shift invariant.

2. Let \( \{X_n, n \in \mathbb{N}\} \) be a Bernoulli process with success probability \( p \) in a single trial. Let \( \{N_n, n \in \mathbb{N}\} \) be the number of successes in first \( n \) trials.

(a) Compute the following probability \( P\{N_5 = 4, N_7 = 5, N_{13} = 8\} \).

(b) Compute the expectation \( E[N_5N_8] \).

**Solution:**

(a) From independent increment property of discrete counting process \( N \) it follows that
\[
P\{N_5 = 4, N_7 = 5, N_{13} = 8\} = P\{N_5 = 4\}P\{N_7 - N_5 = 1\}P\{N_{13} - N_7 = 3\}.
\]

From the stationary increment property it follows that RHS is equal to
\[
P\{N_5 = 4\}P\{N_2 = 1\}P\{N_6 = 3\} = \binom{5}{4}p^4(1-p)\binom{2}{1}p(1-p)\binom{6}{3}p^3(1-p)^3 = 200 \times p^8(1-p)^5.
\]

(b) From independent and stationary increment property of discrete counting process \( N \) it follows
\[
E[N_5N_8] = E[N_5(N_8 - N_5 + N_5)] = EN_3N_3 + EN_5^2.
\]

3. Consider a simple counting process \( \{N_t, t \geq 0\} \) with independent and stationary increments.

(a) Show that \( P\{N_t = 0\} = e^{-\lambda t} \) for some constant \( \lambda \geq 0 \).
(b) Let $G(t) = E \alpha^N$, then show that $G(t) = e^{-\lambda t (1 - \alpha)}$.

(c) Show that $P \{ N_i = n \} = \frac{e^{-\lambda t \alpha^n}}{n!}$.

(d) Compute $\text{Var} N_i$.

**Solution:**

(a) From independent and stationary increment property of simple counting process $N_i$ it follows that

$$g(t+s) \triangleq P \{ N_{i+s} = 0 \} = P \{ N_i = 0 \} P \{ N_s = 0 \} = g(t) g(s).$$

Only continuous $g \neq 0$ that satisfies this semi-group property is $g(t) = e^{-\lambda t}$ for some $\lambda \geq 0$.

(b) Using similar argument as before we can see that

$$G(t+s) \triangleq E \alpha^{N_{i+s}} = E \alpha^N E \alpha^{(N_{i+s} - N_i)} = G(t) G(s).$$

Clearly, $G(t) = e^{g(\alpha)}$, where $G'(0) = g(\alpha)$ and hence we can compute

$$g(\alpha) = \lim_{h \downarrow 0} \frac{G(h) - G(0)}{h} = \lim_{h \downarrow 0} \frac{\sum_{n \in \mathbb{N}_0} \alpha^n P \{ N_h = n \} - 1}{h} = - (1 - \alpha) \lim_{h \downarrow 0} \frac{P \{ N_h \geq 1 \}}{h} = -(1 - \alpha) \lambda.$$

(c) We can expand $G(t) = e^{-(1 - \alpha) \lambda t}$ as a polynomial in $\alpha$ to get the result by comparing coefficients. That is,

$$G(t) = e^{-\lambda t} \sum_{n \in \mathbb{N}_0} \alpha^n \frac{(\lambda t)^n}{n!} = \sum_{n \in \mathbb{N}_0} \alpha^n P \{ N_i = n \}.$$

(d) To compute the variance, we use the probability mass function on discrete valued random variable $N_i$.

$$\text{Var} N_i = E N_i (N_i - 1) + E N_i - (E N_i)^2 = e^{-\lambda t} (\lambda t)^2 \sum_{n \in \mathbb{N}_0} n(n - 1) \frac{(\lambda t)^{n-2}}{n!} + \lambda t - (\lambda t)^2 = \lambda t.$$  

4. Let $S_n$ be the $n$th jump instant of a Poisson process.

(a) Compute $\sum_{n \in \mathbb{N}} P \{ S_n \leq t \}$.

(b) Compute $E e^{-\theta S_n}$.

(c) Compute $E \sum_{n \in \mathbb{N}} f(S_n)$ for any non-negative function on $\mathbb{R}_+$.

**Solution:**

(a)

$$\sum_{n \in \mathbb{N}} P \{ S_n \leq t \} = \sum_{n \in \mathbb{N}} P \{ N_i \geq n \} = E [N_i] = \lambda t.$$

(b) Since the inter-arrival times of a Poisson process are iid and exponentially distributed it follows that for some $\lambda > 0$, we have

$$E [e^{-\theta S_n}] = E [e^{-\theta X}]^n = \left( \frac{\lambda}{\theta + \lambda} \right)^n.$$

(c) Let $f$ be any non-negative function on $\mathbb{R}_+$. Then, using the density of $S_n$ and exchanging infinite summation and integration using Monotone convergence theorem, we get

$$E \sum_{n \in \mathbb{N}} f(S_n) = \sum_{n \in \mathbb{N}} \int_{\mathbb{R}_+} f(t) dF_{S_n}(t) = \int_{\mathbb{R}_+} f(t) d(\sum_{n \in \mathbb{N}} F_{S_n}(t)) = \lambda \int_{\mathbb{R}_+} f(t) dt.$$