1. Let $X_1, X_2, \ldots$ be an i.i.d sequence of random variables. Assume that each $X_i > 0$ almost surely, and that both $\mathbb{E}[X_i]$ and $\mathbb{E}[(\frac{1}{X_i})]$ are finite. Set $S_n = X_1 + \cdots + X_n$. Show that $\mathbb{E}[(\frac{S_n}{S_m})] = 1 + (m - n)\mathbb{E}[X_1]\mathbb{E}[(\frac{1}{X_m})]$ when $m > n$.

2. A light bulb has a lifetime that is exponential with a mean of $\mu$. If $h$ is the time that both a janitor replaces it immediately. In addition, there is a handyman who comes independently of bulb failure, on average, $h = 3$ times per year according to a Poisson process and replaces the light bulb as "preventive maintenance”.
   
   (a) Find the average time between bulb replacements, in terms of $\mu$ and $h$.
   
   (b) In the long run, what fraction of the replacements are due to failure? Your answer should be an expression in terms of $\mu$ and $h$.

3. Let $f : R \rightarrow R$ be a non-decreasing function, i.e., $f(x) \leq f(y)$ if $x \leq y$. Show that $f$ is Borel measurable.
   
   Hint: Suffice to check if $f^{-1}((-\infty,y])$ is Borel measurable.

4. If $S, T$ are stopping times and $S \leq T$ a.s., then show that $\mathcal{F}_S \subseteq \mathcal{F}_T$.

5. Using the definition of conditional expectation, solve the following:
   
   (a) Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $Y$ be a $\mathcal{F}$-measurable random variable with $\mathbb{E}|Y| < \infty$. Let $\mathcal{G}_1 \subseteq \mathcal{G}_2 \subseteq \mathcal{F}$. Show that $\mathbb{E}[Y | \mathcal{G}_1] = \mathbb{E}[\mathbb{E}[Y | \mathcal{G}_2] | \mathcal{G}_1]$.
   
   (b) For any $B \subseteq \mathcal{G}_1$, show that $\mathbb{E}(Y 1_B | \mathcal{G}_1) = 1_B \mathbb{E}(Y | \mathcal{G}_1)$.
   
   (c) (Conditional Jensen’s Inequality) Let $\phi : (a, b) \rightarrow R$ be a differentiable convex function for some $-\infty \leq a < b \leq \infty$. Let $Y$ be a random variable on a probability space $(\Omega, \mathcal{F}, P)$ such that $P(Y \in (a, b)) = 1$ and $\mathbb{E}|\phi(Y)| < \infty$. Let $\mathcal{G}$ be a sub-$\sigma$-algebra of $\mathcal{F}$. Then prove that $\phi(\mathbb{E}(Y | \mathcal{G})) \leq \mathbb{E}(\phi(Y) | \mathcal{G})$.
   
   Hint: for any $c, x \in (a, b)$, $\phi(x) - \phi(c) \geq \phi'(c)(x - c)$.

6. Let $S_n$ be the $n$th jump instant of a renewal process with iid inter-renewal time $X_n$ and renewal function $m(t)$.
   
   (a) Compute $\sum_{n \in \mathbb{N}} P\{S_n \leq t\}$.
   
   (b) Compute $\mathbb{E}e^{-\theta S_n}$.
   
   (c) Compute $\mathbb{E} \sum_{n \in \mathbb{N}} f(S_n)$ for any non-negative function on $\mathbb{R}_+$.

7. Let $S_n$ be the $n$th jump instant of a Poisson process with rate $\lambda$. Compute the following expression for some $t > 0$

   $$\mathbb{E}\left[\sum_{n \in \mathbb{N}} 2S_n 1\{S_n \leq t\} + \sum_{n \in \mathbb{N}} \frac{1}{S_n} 1\{S_n > t\}\right].$$