1. Let $X_1, X_2, \ldots$ be an i.i.d sequence of random variables. Assume that each $X_i > 0$ almost surely, and that both $\mathbb{E}[X_i]$ and $\mathbb{E}[\frac{1}{X_i}]$ are finite. Set $S_n = X_1 + \cdots + X_n$. Show that $\mathbb{E}[\frac{S_n}{X_1}] = 1 + (m - n)\mathbb{E}[X_1]\mathbb{E}[\frac{1}{X_n}]$ when $m > n$.

**Solution:** For $m > n$, we have
\[
\mathbb{E}\left[\frac{S_m}{S_n}\right] = \mathbb{E}\left[1 + \frac{X_{n+1} + \cdots + X_m}{S_n}\right]
= 1 + \sum_{i=n+1}^{m} \mathbb{E}\left[\frac{X_i}{S_n}\right]
= 1 + \sum_{i=n+1}^{m} \mathbb{E}[X_i]\mathbb{E}\left[\frac{1}{S_n}\right]
= 1 + (m - n)\mathbb{E}[X_i]\mathbb{E}\left[\frac{1}{S_n}\right]
\]
where the second last inequality follows since $S_n$ is independent of $X_{n+1}, \ldots, X_m$.

2. A light bulb has a lifetime that is exponential with a mean of $\mu = 200$ days. When it burns out, a janitor replaces it immediately. In addition, there is a handyman who comes independently of bulb failure, on average, $h = 3$ times per year according to a Poisson process and replaces the light bulb as "preventive maintenance".

(a) Find the average time between bulb replacements, in terms of $\mu$ and $h$.

(b) In the long run, what fraction of the replacements are due to failure? Your answer should be an expression in terms of $\mu$ and $h$.

**Solution:**

(a) Let $N_1(t)$ be burnout events, and $N_2(t)$ be handyman replacements. Then $N_3(t) = N_1(t) + N_2(t)$ is a Poisson process with rate $\frac{h}{365} + \frac{1}{\mu} = 0.0132$ times per day. Thus, average time between replacements is $\frac{1}{\frac{h}{365} + \frac{1}{\mu}} \approx 75.7$ days.

(b) The long run fraction of the replacements due to failure, $\lim_{t \to \infty} \frac{N_1(t)}{N_3(t)} = \lim_{t \to \infty} \frac{N_1(t)}{\lim_{t \to \infty} N_3(t)} = \frac{\frac{1}{\mu}}{\frac{h}{365} + \frac{1}{\mu}} = 0.38$ w.p. 1.

3. Let $f : R \to R$ be a non-decreasing function, i.e., $f(x) \leq f(y)$ if $x \leq y$. Show that $f$ is Borel measurable.

**Hint:** Suffice to check if $f^{-1}((-(\infty, y])]$ is Borel measurable.

**Solution:** Let $a, b \in R$, $a \leq b$ and $b \in f^{-1}((-(\infty, y])]$. If $a \notin f^{-1}((-(\infty, y])]$ then $f(a) > f(b)$ which is a contradiction. Hence, for every $b \in f^{-1}((-(\infty, y])]$, $(-(\infty, b] \subseteq f^{-1}((-(\infty, y])]$. Thus, $f^{-1}((-(\infty, y])]$ can only be subsets of the form $\emptyset, R, (-(\infty, b]$ or $(-(\infty, b)$ for some $b \in R$, each of which are Borel sets. Hence, $f$ is Borel measurable.
5. Using the definition of conditional expectation, solve the following:

(a) Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $Y$ be a $\mathcal{F}$-measurable random variable with $E|Y| < \infty$. Let $\mathcal{G}_1 \subset \mathcal{G}_2 \subset \mathcal{F}$. Show that $E[Y|\mathcal{G}_1] = E[E[Y|\mathcal{G}_2]|\mathcal{G}_1]$.

(b) For any $B \in \mathcal{G}_1$, show that $E(Y1_B|\mathcal{G}_1) = 1_B E(Y|\mathcal{G}_1)$.

(c) (Conditional Jensen’s Inequality) Let $\phi : (a, b) \to \mathbb{R}$ be a differentiable convex function for some $-\infty \leq a < b \leq \infty$. Let $Y$ be a random variable on a probability space $(\Omega, \mathcal{F}, P)$ such that $P(Y \in (a, b)) = 1$ and $E|\phi(Y)| < \infty$. Let $\mathcal{G}$ be a sub-$\sigma$-algebra of $\mathcal{F}$. Then prove that $\phi(E(Y|\mathcal{G})) \leq E(\phi(Y)|\mathcal{G})$.

**Solution:**

(a) Let $A \in \mathcal{G}_1$, $Z_1 = E(Y|G_1)$ and $Z_2 = E(Y|G_2)$. Then $E(Y1_A) = E(Z_11_A)$ by the definition of $Z_1$. Since $G_1 \subset G_2$, $A \in G_2$ and by the definition of $Z_2$, $E(Y1_A) = E(Z_21_A)$. Thus, $E(Z_11_A) = E(Z_21_A)$ for all $A \in \mathcal{G}_1$. Then by the definition of $E(Z_2|\mathcal{G}_1)$, it follows that $Z_1 = E(Z_2|G_1)$, proving the statement.

(b) Given $B \in \mathcal{G}_1$, for any $A \in \mathcal{G}_1$, $A \cap B \in \mathcal{G}_1$ and by definition, $E[Y1_B1_A] = E[Y1_{A \cap B}] = E(Z_11_{A \cap B}) = E(Z_11_B \cdot 1_A)$. So $E(Y1_B|\mathcal{G}_1) = Z_11_B$.

(c) Taking $c = E(Y|\mathcal{G})$ and $x = Y$ gives, $\phi(Y) - \phi(E(Y|\mathcal{G})) \geq \phi'(E(Y|\mathcal{G})) (Y - E(Y|\mathcal{G}))$. Applying conditional expectation w.r.t $\mathcal{G}$ gives

$$E[\phi(Y)|\mathcal{G}] - E[\phi(E(Y|\mathcal{G})|\mathcal{G})] \geq E[\phi'(E(Y|\mathcal{G})) (Y - E(Y|\mathcal{G}))|\mathcal{G}]$$

Since, $E(Y|\mathcal{G}) \in \mathcal{G}$ any measurable function of $E(Y|\mathcal{G})$ is also contained in $\mathcal{G}$. Thus,

$$E[\phi(Y)|\mathcal{G}] - \phi(E(Y|\mathcal{G})) \geq \phi'(E(Y|\mathcal{G})) E[(Y - E(Y|\mathcal{G}))|\mathcal{G}]$$

$$= \phi'(E(Y|\mathcal{G})) [E[Y|\mathcal{G}] - E(Y|\mathcal{G})] = 0.$$

which gives $E[\phi(Y)|\mathcal{G}] \geq \phi(E(Y|\mathcal{G}))$.

6. Let $S_n$ be the $n$th jump instant of a renewal process with $iid$ inter-renewal time $X_n$ and renewal function $m(t)$.

(a) Compute $\sum_{n \in \mathbb{N}} P\{S_n \leq t\}$.

(b) Compute $Ee^{-\theta S_n}$.

(c) Compute $E \sum_{n \in \mathbb{N}} f(S_n)$ for any non-negative function on $\mathbb{R}_+$.

**Solution:**
(a)\[\sum_{n \in \mathbb{N}} P\{S_n \leq t\} = \sum_{n \in \mathbb{N}} P\{N(t) \geq n\} = \mathbb{E}[N(t)] = m(t).\]

(b) Since the inter-arrival times of a Poisson process are iid, it follows that for some $\lambda > 0$, we have
\[\mathbb{E}[e^{-\theta S_n}] = \mathbb{E}[e^{-\theta X}] = M_X(\theta)^n.\]

(c) Let $f$ be any non-negative function on $\mathbb{R}_+$. Then, using the density of $S_n$ and exchanging infinite summation and integration using Monotone convergence theorem, we get
\[\mathbb{E}\sum_{n \in \mathbb{N}} f(S_n) = \sum_{n \in \mathbb{N}} \int_{\mathbb{R}_+} f(t) dF_{S_n}(t) = \int_{\mathbb{R}_+} f(t) d(\sum_{n \in \mathbb{N}} F_{S_n}(t)) = \int_{\mathbb{R}_+} f(t) dm(t).\]

7. Let $S_n$ be the $n$th jump instant of a Poisson process with rate $\lambda$. Compute the following expression for some $t > 0$
\[\mathbb{E}\left[\sum_{n \in \mathbb{N}} 2S_n 1\{S_n \leq t\} + \sum_{n \in \mathbb{N}} \frac{1}{S_n^2} 1\{S_n > t\}\right].\]

**Solution:** Since we know that $\mathbb{E}\sum_{n \in \mathbb{N}} f(S_n) = \lambda \int_{\mathbb{R}_+} f(t) dt$, we get
\[\mathbb{E}\left[\sum_{n \in \mathbb{N}} 2S_n 1\{S_n \leq t\} + \sum_{n \in \mathbb{N}} \frac{1}{S_n^2} 1\{S_n > t\}\right] = \lambda \left( \int_0^t 2x dx + \int_t^\infty \frac{dx}{x^2} \right) = \lambda (t^2 + 1/t).\]