1. Let $(\Omega, \mathcal{F}, P)$ be a probability space, and let $(X_n)_{n \in \mathbb{N}}$ be a collection of mutually independent $\mathcal{F}$-measurable random variables, with

$$P\left(X_n = \frac{1}{2^n}\right) = \frac{1}{2} = P\left(X_n = -\frac{1}{2^n}\right), \quad n \in \mathbb{N}.$$ 

(a) Does the above sequence converge to 0 in the mean-squared sense? Justify your answer. 

(b) Does the above sequence converge to 0 in the almost sure sense? Justify your answer. 

Solution:

(a) We note that for each $n \in \mathbb{N}$, $|X_n| = \frac{1}{2^n}$ with probability 1. Hence,

$$E[(X_n - 0)^2] = \frac{1}{4^n} \rightarrow 0 \text{ as } n \rightarrow \infty,$$

thereby implying that $X_n$ converges to 0 in the mean-squared sense.

(b) For any $\epsilon > 0$, we have

$$P(|X_n - 0| > \epsilon) \leq \frac{E[X_n^2]}{\epsilon^2} = \frac{1}{4^n \epsilon^2},$$

where the first line above is due to Chebyshev’s inequality. Summing both sides over all $n \in \mathbb{N}$, we get

$$\sum_{n \in \mathbb{N}} P(|X_n - 0| > \epsilon) \leq \sum_{n \in \mathbb{N}} \frac{1}{4^n \epsilon^2} < \infty$$

for every $\epsilon > 0$. Thus, by the first part of Borel-Cantelli lemma, we get

$$P(|X_n| > \epsilon \text{ i.o.}) = 0 \text{ for every } \epsilon > 0.$$

This then implies that $X_n$ converges to 0 almost surely.