1. Let \((\Omega, \mathcal{F}, P)\) be a probability space, and let \(X\) and \(Y\) be two independent \(\mathcal{F}\)-measurable geometric random variables with parameters \(p\) and \(q\) respectively, where \(0 < p < 1\) and \(0 < q < 1\). That is,

\[
P(X = k) = p(1-p)^{k-1}, \quad k \in \{1,2,\ldots\},
\]

\[
P(Y = k) = q(1-q)^{k-1}, \quad k \in \{1,2,\ldots\}.
\]

(a) (Memoryless property of geometric distribution)

Show that the relation

\[
P(X > k_1 + k_2 | X > k_1) = P(X > k_2).
\]

holds for any \(k_1 \in \{1,2,\ldots\}\) and \(k_2 \in \{1,2,\ldots\}\). (2)

(b) Define a new \(\mathcal{F}\)-measurable random variable \(Z\) as

\[
Z(\omega) = \min\{X(\omega), Y(\omega)\}, \quad \omega \in \Omega.
\]

Write down the distribution of \(Z\) and compute \(E[Z]\). (3)

**Solution:**

(a) We compute the distribution of \(Z\) by conditioning on \(X_1\). First, we note that \(Z\) takes values in the set \(A := \{1,2,\ldots\}\). Further, by the law of total probability, for any \(k \in A\), we have

\[
P(Z = k) = \sum_{k_1=1}^{\infty} P(Z = k | X_1 = k_1) \cdot P(X_1 = k_1).
\]

We now observe that

\[
P(Z = k | X_1 = k_1) = P(\min(k_1, X_2) = k | X_1 = k_1) = \begin{cases} P(X_2 = k), & k_1 > k, \\ P(X_2 \geq k), & k_1 = k, \\ 0, & k_1 < k. \end{cases}
\]

Thus, we have

\[
P(Z = k) = P(X_2 \geq k) \cdot P(X_1 = k) + P(X_2 = k) \cdot \left(\sum_{k_1=k+1}^{\infty} P(X_1 = k_1)\right)
\]

\[
= P(X_2 > k) \cdot P(X_1 = k) + P(X_2 = k) \cdot \left(\sum_{k_1=k}^{\infty} P(X_1 = k_1)\right).
\]

The first term on the right hand side is given by

\[
P(X_2 > k) \cdot P(X_1 = k) = p(1-p)^{k-1} \cdot \sum_{n=k}^{\infty} q(1-q)^{n-1} = p(1-p)^{k-1}(1-q)^{k-1},
\]

while the second term is given by

\[
P(X_2 = k) \cdot \left(\sum_{k_1=k}^{\infty} P(X_1 = k_1)\right) = q(1-q)^{k-1} \cdot \left(\sum_{k_1=k}^{\infty} p(1-p)^{k_1-1}\right) = q(1-q)^{k-1}(1-p)^{k-1}.
\]
Combining, we get
\[ P(Z = k) = p(1 - p)^{k-1}(1 - q)^k + q(1 - q)^{k-1}(1 - p)^k = (p + q - pq)(1 - (p + q - pq))^{k-1}. \]

Thus, Z is a geometric random variable with parameter \( p = p + q - pq \). We compute the distribution of Z by conditioning on \( X_1 \). First, we note that Z takes values in the set \( A := \{1, 2, \ldots \} \). Further, by the law of total probability, for any \( k \in A \), we have

\[ P(Z = k) = \sum_{k_1=1}^{\infty} P(Z = k | X_1 = k_1) \cdot P(X_1 = k_1). \]

We now observe that

\[ P(Z = k | X_1 = k_1) = P(\min(k_1, X_2) = k | X_1 = k_1) = \begin{cases} P(X_2 = k), & k_1 > k, \\ P(X_2 \geq k), & k_1 = k, \\ 0, & k_1 < k. \end{cases} \]

Thus, we have

\[ P(Z = k) = P(X_2 \geq k) \cdot P(X_1 = k) + P(X_2 = k) \cdot \left( \sum_{k_1=k+1}^{\infty} P(X_1 = k_1) \right). \]

The first term on the right hand side is given by

\[ P(X_2 > k) \cdot P(X_1 = k) = p(1 - p)^{k-1} \cdot \sum_{n=k+1}^{\infty} q(1 - q)^{n-1} = p(1 - p)^{k-1}(1 - q)^k, \]

while the second term is given by

\[ P(X_2 = k) \cdot \left( \sum_{k_1=k}^{\infty} P(X_1 = k_1) \right) = q(1 - q)^{k-1} \cdot \left( \sum_{k_1=k}^{\infty} p(1 - p)^{k_1-1} \right) = q(1 - q)^{k-1}(1 - p)^{k-1}. \]

Combining, we get

\[ P(Z = k) = p(1 - p)^{k-1}(1 - q)^k + q(1 - q)^{k-1}(1 - p)^k = (p + q - pq)(1 - (p + q - pq))^{k-1}. \]

Thus, Z is a geometric random variable with parameter \( \nu = p + q - pq \).

Next, we compute the mean of Z as

\[ E[Z] = \sum_{k=1}^{\infty} k \cdot P(Z = k) = \nu \sum_{k=1}^{\infty} k(1 - \nu)^{k-1} = \frac{1}{\nu}. \]

(b) We have

\[ P(X > k_1 + k_2) = \sum_{k=k_1+k_2+1}^{\infty} P(X = k) = \sum_{k=k_1+k_2+1}^{\infty} p(1 - p)^{k-1} = (1 - p)^{k_1+k_2} \]

\[ = (1 - p)^{k_1} \cdot (1 - p)^{k_2} = \left( \sum_{k=k_1+1}^{\infty} p(1 - p)^{k} \right) \left( \sum_{k=k_2+1}^{\infty} p(1 - p)^{k} \right) = P(X > k_1) \cdot P(X > k_2). \]

Dividing both sides of the above equation by \( P(X_1 > k_1) \), we get the desired result.