1. Let \((\Omega, \mathcal{F}, P)\) be a probability space, and let \(X\) and \(Y\) be two independent \(\mathcal{F}\)-measurable geometric random variables with parameters \(p\) and \(q\) respectively, where \(0 < p < 1\) and \(0 < q < 1\). That is,

\[
P(X = k) = p(1-p)^{k-1}, \quad k \in \{1, 2, \ldots\},
\]

\[
P(Y = k) = q(1-q)^{k-1}, \quad k \in \{1, 2, \ldots\}.
\]

(a) (Memoryless property of geometric distribution)
Show that the relation

\[
P(X > k_1 + k_2 | X > k_1) = P(X > k_2).
\]

holds for any \(k_1 \in \{1, 2, \ldots\}\) and \(k_2 \in \{1, 2, \ldots\}\). (2)

(b) Define a new \(\mathcal{F}\)-measurable random variable \(Z\) as

\[
Z(\omega) = \min\{X(\omega), Y(\omega)\}, \quad \omega \in \Omega.
\]

Write down the distribution of \(Z\) and compute \(E[Z]\). (3)

Solution:
(a) We have

\[
P(X > k_1 + k_2) = (1-p)^{k_1 + k_2}
\]

\[
= (1-p)^{k_1}(1-p)^{k_2}
\]

\[
= P(X > k_1) \cdot P(X > k_2).
\]

Dividing both sides of the above equation by \(P(X > k_1)\), we get the desired result.

(b) We note that

\[
P(X > k) = (1-p)^k, \quad P(Y > k) = (1-q)^k
\]

for all \(k \in \{1, 2, \ldots\}\). Using this, we have

\[
P(Z > k) = P(\min\{X, Y\} > k)
\]

\[
\overset{(a)}{=} P(X > k, Y > k)
\]

\[
\overset{(b)}{=} P(X > k) \cdot P(Y > k)
\]

\[
= ((1-p)(1-q))^k
\]

\[
= (1-(p+q-pq))^k,
\]

where (a) follows by noting that minimum of two numbers is greater than \(k\) if and only if each of the two numbers is greater than \(k\), and (b) follows from independence of \(X\) and \(Y\). Therefore, it follows from the above set of equations that \(Z\) is geometric with parameter \(\nu = p + q - pq\). Then, we have

\[
E[Z] = \sum_{k=1}^{\infty} k \cdot P(Z = k) = \sum_{k=1}^{\infty} k \cdot \nu \cdot (1-\nu)^{k-1} = \frac{1}{\nu}.
\]