Lecture-27: Poisson Process

1 Poisson and exponential random variables

A whole number valued random variable \(N \in \mathbb{N}_0\) is called Poisson if for some constant \(\lambda > 0\), we have

\[
P(N = n) = e^{-\lambda} \frac{\lambda^n}{n!}.
\]

It is easy to check that \(\mathbb{E}N = \text{Var}N = \lambda\). Furthermore, the moment generating function \(M_N(t) = \mathbb{E}e^{tN} = e^{\lambda(e^t-1)}\) exists for all \(t \in \mathbb{R}\).

1.1 Memoryless distribution

A random variable \(X\) with continuous support on \(\mathbb{R}^+\), is called memoryless if

\[
P\{X > s\} = P\{X > t+s|X > t\}\quad \text{for all } t, s \in \mathbb{R}^+.
\]

**Proposition 1.1.** The unique memoryless distribution function with continuous support on \(\mathbb{R}^+\) is the exponential distribution.

**Proof.** Let \(X\) be a random variable with a memoryless distribution function \(F: \mathbb{R}^+ \rightarrow [0, 1]\). It follows that \(\bar{F}(t) \triangleq 1 - F(t)\) satisfies the semi-group property

\[
\bar{F}(t + s) = \bar{F}(t)\bar{F}(s).
\]

Since \(\bar{F}(x) = P\{X > x\}\) is non-increasing in \(x \in \mathbb{R}^+\), we have \(\bar{F}(x) = e^{\theta x}\), for some \(\theta < 0\) from Lemma A.1.

2 Simple point processes

A simple point process is a collection of distinct points \(\Phi = \{S_n \in \mathbb{R}^d: n \in \mathbb{N}\}\), such that \(|S_n| \rightarrow \infty\) as \(n \rightarrow \infty\).

Let \(N(\emptyset) = 0\) and denote the number of points in a set \(A \subseteq \mathbb{R}^d\) by \(N(A) = \sum_{n \in \mathbb{N}} 1\{S_n \in A\}\). Then \((N(A): A \in \mathcal{F})\) is called a counting process for the point process \(\Phi\). A counting process is simple if the underlying process is simple.

Point processes can model many interesting physical processes.

1. Arrivals at classrooms, banks, hospital, supermarket, traffic intersections, airports etc.
2. Location of nodes in a network, such as cellular networks, sensor networks, etc.

2.1 Simple point processes in one-dimension

We can simplify this definition for \(d = 1\). In \(\mathbb{R}^+\), one can order the points \((S_n: n \in \mathbb{N})\) of the point process \(\Phi\). The number of points in the interval \((0, t]\) is \(N((0, t]) = \sum_{n \in \mathbb{N}} 1\{S_n \in (0, t]\}\) as denoted by \(N(t)\). For \(s < t\), the number of points in interval \((s, t]\) is \(N((s, t]) = N((0, t]) - N((0, s]) = N(t) - N(s)\).

A stochastic process \((N(t) : t \geq 0)\) is a counting process if

1. \(N(0) = 0\), and
2. for each \(\omega \in \Omega\), the map \(t \mapsto N(t)\) is non-decreasing, integer valued, and right continuous.
Lemma 3.1. For any finite time $t$ for $n$\ for it is easy to see that
\begin{equation*}
\text{Proof. (the inter-arrival times (A simple counting process has the unit jump size almost surely. General point processes in higher dimension don't have any inter-arrival time interpretation.}
\end{equation*}

Lemma 2.4. Let $F_n$ be the distribution function for $S_n$, then $P_n(t) \triangleq P\{N(t) = n\} = F_n(t) - F_{n+1}(t)$.
\begin{proof}
It suffices to observe that following is a union of disjoint events,
\begin{equation*}
\{S_n \leq t\} = \{S_n \leq t, S_{n+1} > t\} \cup \{S_n \leq t, S_{n+1} \leq t\}.
\end{equation*}
\end{proof}

\section{Poisson process}
A simple counting process $(N(t) : t \geq 0)$ is called a homogeneous Poisson process with a finite positive rate $\lambda$, if the inter-arrival times $(X_n : n \in \mathbb{N})$ are iid random variables with an exponential distribution of rate $\lambda$. That is, it has a distribution function $F : \mathbb{R}_+ \to [0, 1]$, such that $F(x) = 1 - e^{-\lambda x}$ for all $x \in \mathbb{R}_+$.

For many proofs regarding Poisson processes, we partition the sample space with the disjoint events $\{N(t) = n\}$ for $n \in \mathbb{N}_0$. We need the following lemma that enables us to do that.

Lemma 3.1. For any finite time $t > 0$, a Poisson process is finite almost surely.
Proof. By strong law of large numbers, we have
\[
\lim_{n \to \infty} \frac{S_n}{n} = E[X_1] = \frac{1}{\lambda} \quad \text{a.s.}
\]
Fix \( t > 0 \) and we define a sample space subset \( M = \{ \omega \in \Omega \mid N(\omega, t) = \infty \} \). For any \( \omega \in M \), we have \( S_n(\omega) \leq t \) for all \( n \in \mathbb{N} \). This implies \( \limsup_n \frac{S_n}{n} = 0 \) and \( \omega \notin \{ \lim_n \frac{S_n}{n} = \frac{1}{\lambda} \} \). Hence, the probability measure for set \( M \) is zero. \( \square \)

3.1 Distribution functions

Lemma 3.2. Moment generating function of arrival times \( S_n \) is
\[
M_{S_n}(t) = E[e^{S_n}] = \begin{cases} 
\frac{e^{\lambda t}}{(\lambda - t)^n}, & t < \lambda \\
\infty, & t \geq \lambda.
\end{cases}
\]

Lemma 3.3. Distribution function of \( S_n \) is given by \( F_n(t) \triangleq P\{S_n \leq t\} = 1 - e^{-\lambda t} \sum_{k=0}^{n-1} \frac{(\lambda t)^k}{k!} \).

Theorem 3.4. Density function of \( S_n \) is Gamma distributed with parameters \( n \) and \( \lambda \). That is,
\[
f_n(s) = \frac{\lambda (\lambda s)^{n-1}}{(n-1)!} e^{-\lambda s}.
\]

Theorem 3.5. For each \( t > 0 \), the distribution of Poisson process \( N(t) \) with parameter \( \lambda \) is given by
\[
P\{N(t) = n\} = e^{-\lambda t} \frac{(\lambda t)^n}{n!}.
\]
Further, \( E[N(t)] = \lambda t \), explaining the rate parameter \( \lambda \) for Poisson process.

Proof. Result follows from density of \( S_n \) and recognizing that \( P_n(t) = F_n(t) - F_{n+1}(t) \). \( \square \)

Corollary 3.6. Distribution of arrival times \( S_n \) is
\[
F_n(t) = \sum_{j \geq n} P_j(t), \quad \sum_{n \in \mathbb{N}} F_n(t) = E[N(t)].
\]

Proof. First result follows from the telescopic sum and the second from the following observation.
\[
\sum_{n \in \mathbb{N}} F_n(t) = \mathbb{E} \sum_{n \in \mathbb{N}} 1\{N(t) \geq n\} = \sum_{n \in \mathbb{N}} P\{N(t) \geq n\} = \mathbb{E}[N(t)].
\]

A Poisson process is not a stationary process. That is, the finite dimensional distributions are not shift invariant. This is clear from looking at the first moment \( \mathbb{E}[N(t)] = \lambda t \), which is linearly increasing in time.

A Functions with semigroup property

Lemma A.1. A unique non-negative right continuous function \( f : \mathbb{R} \to \mathbb{R} \) satisfying the semigroup property
\[
f(t + s) = f(t)f(s), \quad \text{for all } t, s \in \mathbb{R}
\]
is \( f(t) = e^{\theta t} \), where \( \theta = \log f(1) \).

Proof. Clearly, we have \( f(0) = f^2(0) \). Since \( f \) is non-negative, it means \( f(0) = 1 \). By definition of \( \theta \) and induction for \( m, n \in \mathbb{Z}^+ \), we see that
\[
f(m) = f(1)^m = e^{\theta m}, \quad e^{\theta} = f(1) = f(1/n)^n.
\]
Let \( q \in \mathbb{Q} \), then it can be written as \( m/n, n \neq 0 \) for some \( m, n \in \mathbb{Z}^+ \). Hence, it is clear that for all \( q \in \mathbb{Q}^+ \), we have \( f(q) = e^{\theta q} \), either unity or zero. Note, that \( f \) is a right continuous function and is non-negative. Now, we can show that \( f \) is exponential for any real positive \( t \) by taking a sequence of rational numbers \( (q_n : n \in \mathbb{N}) \) decreasing to \( t \). From right continuity of \( f \), we obtain
\[
f(t) = \lim_{q_n \downarrow t} f(q_n) = \lim_{q_n \downarrow t} e^{\theta q_n} = e^{\theta t}.
\]
\( \square \)