Lecture-20: Stopping Time \(\sigma\)-algebra

1 Wald’s Lemma

Lemma 1.1 (Wald’s Lemma). Consider a random walk \((S_n : n \in \mathbb{N})\) with iid step-sizes \((X_n : n \in \mathbb{N})\) having finite \(\mathbb{E}[X_1]\). Let \(N\) be a finite mean stopping time adapted to the natural filtration \(\mathcal{F}_n = (\mathcal{F}_n = \sigma(X_1, \ldots, X_n) : n \in \mathbb{N})\). Then,

\[
\mathbb{E}S_N = \mathbb{E}X_1\mathbb{E}N.
\]

Proof. From the independence of step sizes, it follows that \(X_n\) is independent of \(\mathcal{F}_{n-1}\). Next we observe that \(\{N \geq n\} = \{N > n - 1\} \in \mathcal{F}_{n-1}\), and hence \(\mathbb{E}[X_n 1_{\{N > n\}}] = \mathbb{E}X_n \mathbb{E}[1_{\{N > n\}}]\). Therefore,

\[
\mathbb{E}\sum_{n=1}^N X_n = \mathbb{E}\sum_{n=1}^N X_n 1_{\{N \geq n\}} = \sum_{n=1}^N \mathbb{E}X_n \mathbb{E}[1_{\{N \geq n\}}] = \mathbb{E}X_1 \mathbb{E}\left[\sum_{n=1}^N 1_{\{N \geq n\}}\right] = \mathbb{E}[X_1] \mathbb{E}[N].
\]

We exchanged limit and expectation in the above step, which is not always allowed. We were able to do it since the summand is positive and we apply monotone convergence theorem. \(\square\)

Corollary 1.2. Consider the stopping time \(T_i = \min\{n \in \mathbb{N} : S_n = i\}\) for an integer random walk \(S\) with iid steps \(X\). Then, the mean of stopping time \(\mathbb{E}T_i = i/\mathbb{E}X_1\).

A Wald type result for a random sum \(S_N = \sum_{n=1}^N X_n\) of iid random variables \(X = (X_n : n \in \mathbb{N})\), when \(N\) is independent of the sequence \(X\) is trivial to obtain, since

\[
\mathbb{E}[S_N] = \mathbb{E}[\mathbb{E}[S_N | N]] = \mathbb{E}[N \mathbb{E}X_1] = \mathbb{E}N \mathbb{E}X_1.
\]

When the random variable \(N\) is not independent of the underlying process \(X\), the linearity of expectation of the random sum \(S_N\) does not always hold. For example, let’s take our counting process \((N_n : n \in \mathbb{N})\) for the number of successes in a iid Bernoulli trial. We take the discrete random time \(\tau' = K \wedge \max\{n \in \mathbb{N} : N_n = 1\}\). Then, \(\mathbb{E}N_{\tau'} = 1\), however \(P(\tau' = K) = 1\) and hence \(\mathbb{E}\tau' \mathbb{E}X_1 = Kp \neq 1\) for all \(p \neq 1/K\). However, when the random variable \(N\) is a stopping time with respect to the natural filtration for this process, then even though \(N\) is not independent of the sequence \(X\), the linearity holds. For the same counting process, we can take the stopping time \(\tau = \min\{N_n = 1\}\). Then,

\[
1_{\{\tau = i\}}P(\{X_{[0]} = (0,0,\ldots,1)\}|\sigma(\tau)) = P(\{X_{[0]} = (0,0,\ldots,1)\}|\tau = i) = 1 \neq (q)^{i-1} p = \prod_{j=1}^{i-1} P(X_j = 0) P(X_i = 1).
\]

Time for first success is a geometrically distributed random variable with mean \(1/\mathbb{E}X_1\), hence we can check that \(\mathbb{E}N_\tau = 1 = \mathbb{E}X_1 \mathbb{E}\tau\).

2 Stopping time \(\sigma\)-algebra

We wish to define a \(\sigma\)-algebra consisting information of the process till a random time \(\tau\). For a countable stopping time \(\tau\), what we want is something like \(\sigma(X_1, \ldots, X_\tau)\). But that doesn’t make sense, since the random time is a random variable itself. When \(\tau\) is a stopping time, the event \(\{\tau \leq t\} \in \mathcal{F}_t\). What makes sense is the set of all measurable sets whose intersection with \(\{\tau \leq t\}\) belongs to \(\mathcal{F}_t\) for all \(t \geq 0\).

For a stopping time \(\tau : \Omega \to \mathbb{R}_+\) adapted to the filtration \(\mathcal{F}_\tau\), the stopping time \(\sigma\)-algebra is defined as

\[
\mathcal{F}_\tau = \{A \in \mathcal{F} : A \cap \{\tau \leq t\} \in \mathcal{F}_t, \forall t \geq 0\}.
\]

One can check that \(\mathcal{F}_\tau\) is indeed a \(\sigma\)-algebra. Further, \(\mathcal{F}_\tau\) has information up to the random time \(\tau\). That is, it is a collection of measurable sets that are determined by the process till time \(\tau\). Any measurable set \(A \in \mathcal{F}\) can be written as \(A = (A \cap \{\tau \leq t\}) \cup (A \cap \{\tau > t\})\). All such sets \(A\) such that \(A \cap \{\tau \leq t\} \in \mathcal{F}_t\) is a member of the stopped \(\sigma\)-algebra.
Lemma 2.1. Let $\mathcal{F}_t$ be the natural filtration associated with the process $(X_t : t \in T)$, and $\tau$ be the associated stopping time. Let $Y_t = X_{\tau+t}$, that is $Y_t = X_t1_{\{s \leq t\}} + X_t1_{\{s > t\}}$. Then $\mathcal{F}_\tau = \sigma(Y_s, s \leq t)$.

Proof.

Lemma 2.2. Let $\tau, \tau_1, \tau_2$ be stopping times adapted to a filtration $\mathcal{F}_\tau$. Then, the following are true.

i. $\sigma(\tau) \subseteq \mathcal{F}_\tau$.

ii. If $\tau_1 \leq \tau_2$ almost surely, then $\mathcal{F}_{\tau_1} \subseteq \mathcal{F}_{\tau_2}$.

Proof. Let $\tau$ be a stopping time adapted to a filtration $\mathcal{F}_\tau$. Then, for any $t \geq 0$, we have $\{\tau \leq t\} \in \mathcal{F}_t$.

i. We show that for any $s \geq 0$, the event $\{\tau \leq s\} \in \mathcal{F}_\tau$. This is true because for any $t \geq 0$

$$\{\tau \leq s\} \cap \{\tau \leq t\} = \{\tau \leq s \wedge t\} \in \mathcal{F}_\tau.$$

ii. From the hypothesis, we have $\{\tau_2 \leq t\} \subseteq \{\tau_1 \leq t\}$ almost surely. Let $A \in \mathcal{F}_{\tau_1}$ then $A \cap \{\tau_1 \leq t\} \in \mathcal{F}_t$ for all $t \geq 0$. Further, we see that $A \cap \{\tau_2 \leq t\} = A \cap \{\tau_1 \leq t\} \cap \{\tau_1 \leq t\} \in \mathcal{F}_t$ for all $t \geq 0$.

3 Strong Markov property

Let $X$ be a real valued Markov process adapted to a filtration $\mathcal{F}_\tau$. Let $\tau$ be an almost surely finite stopping time with respect to this filtration, then the process $X$ is called strongly Markov if for all $x \in \mathbb{R}$ and $t > 0$, we have

$$P([X_{t+\tau} \leq x]) = P([X_{t+\tau} \leq x] \mid \sigma(\tau)).$$

Lemma 3.1. Let $(X_t : t \in T)$ be any Markov process adapted to filtration $(\mathcal{F}_t : t \in T)$. For any almost surely finite stopping time $\tau$ with respect to this filtration that takes only countably many values, Markov process $X$ is strongly Markov at this stopping time $\tau$.

Proof. Let $I \subseteq T$ be the countable set such that $\{\tau \in I\} = \Omega$. Let $A \in \mathcal{F}_\tau$, then $A \cap \{\tau = i\} \in \mathcal{F}_t$ for all $i \in I$. Then,

$$\mathbb{E}[A1_{(X_{t+\tau} \leq x)}] = \sum_{i \in I} \mathbb{E}[A \cap (X_{t+\tau} \leq x) \cap \{\tau = i\}] = \sum_{i \in I} \mathbb{E}[\mathbb{E}[A \cap (X_{t+\tau} \leq x) \cap \{\tau = i\} \mid \mathcal{F}_t]] = \sum_{i \in I} \mathbb{E}[\mathbb{E}[A \cap \{\tau = i\} \mid \mathcal{F}_\tau]] \mathbb{E}[1_{(X_{t+\tau} \leq x)} \mid \sigma(\tau))]$$

The result follows since $P([X_{t+\tau} \leq x] \mid \sigma(\tau))) \in \mathcal{F}_\tau$.

Corollary 3.2. Any Markov process on countable index set $T$ is strongly Markov.

Proof. For a countable index set $T$, all associated stopping times assume at most countably many values.

Corollary 3.3. Let $\tau$ be an almost surely finite stopping time with respect to the natural filtration $\mathcal{F}_\tau$ of an iid random sequence $X$. Then $(X_{\tau+1}, \ldots, X_{\tau+n})$ is independent of $\mathcal{F}_\tau$ for each $n \in \mathbb{N}$ and identically distributed to $(X_1, \ldots, X_n)$. 

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