1. Assume that cars arrive as a Poisson process of rate 10 per hour. Assume that each car will pick up a hitchhiker (standing in a queue) independently of the others with probability 1/10. You are second in line. What is the probability that you will have to wait for more than 2 hours?

**Solution:** Suppose $X_1$ denotes the time at which the first hitchhiker gets picked up, and $X_2$ the time at which the second hitchhiker gets picked up. We wish to compute $P(X_2 > 2)$. The occurrence of event $\{X_2 > 2\}$ implies (and is implied by) the occurrence of exactly one of the following events:

(a) $X_1 > 2$

(b) $X_1 \leq 2$ and the second hitchhiker was not picked up in the time interval $(X_1, 2]$.

We now proceed to compute the probability of the event in (a) above. The event that the first hitchhiker waits for more than 2 hours is identical to the event that no car arrived in the first two hours, or none of the cars which arrived in the first two hours picked up the hitchhiker. Thus,

$$P(X_1 > 2) = P(N(2) = 0) + \sum_{k=1}^{\infty} P(N(2) = k) \cdot \left(\frac{9}{10}\right)^k = e^{-2}.$$  

In fact, it can be shown that $P(X_1 > s) = e^{-s}$ for all $s \geq 0$. On similar lines, the probability of the event in (b) may be evaluated as follows:

$$P(X_1 \leq 2, X_2 \text{ is not picked up in } (X_1, 2]) = \int_0^2 \left(\sum_{k=0}^{\infty} P(N(2) = k) \cdot \left(\frac{9}{10}\right)^k\right)e^{-s} ds = 2e^{-2}.$$  

Thus, the desired probability is $P(X_2 > 2) = e^{-2} + 2e^{-2} = 3e^{-2}$.

2. Let $\{N(t) : t \geq 0\}$ be a Poisson process with rate $\lambda$. Let $S_1$ denote the time of first arrival.

(a) Let $t > 0$ be some fixed time instant. Given that $N(t) = 1$, what is the conditional distribution of $S_1$?

(b) More generally, given that $N(t) = n$, where $n \geq 2$, what is the conditional distribution of $S_1$?

**Solution:**

(a) We have

$$P(S_1 > s | N(t) = 1) = \frac{P(S_1 > s, N(t) = 1)}{P(N(t) = 1)}.$$  

If $s > t$, the above quantity is clearly 0. We thus focus on the case when $s \leq t$. We use the fact that $\{S_1 > s, N(t) = 1\} = \{N(s) = 0, N(t) - N(s) = 1\}$ and subsequently the independent increments property to arrive at

$$P(S_1 > s | N(t) = 1) = \frac{P(N(s) = 0) \cdot P(N(t) - N(s) = 1)}{P(N(t) = 1)} = 1 - \frac{s}{t}.$$  

Thus, conditioned on the fact that one arrival has happened in the interval $(0, t]$, the first arrival follows a uniform distribution on $(0, t]$.

(b) Focusing first on the case when $n > 0$, we have

$$P(S_1 > s | N(t) = n) = \frac{P(S_1 > s, N(t) = n)}{P(N(t) = n)}.$$  

If $s > t$, the above quantity is clearly 0. We thus focus on the case when $s \leq t$. We use the fact that $\{S_1 > s, N(t) = n\} = \{N(s) = 0, N(t) - N(s) = n\}$ and subsequently the independent increments property to arrive at

$$P(S_1 > s | N(t) = n) = \frac{P(N(s) = 0) \cdot P(N(t) - N(s) = n)}{P(N(t) = n)} = \left(1 - \frac{s}{t}\right)^n.$$
For the case when $n = 0$, for any $s > t$, we have

$$P(S_1 > s | N(t) = 0) = \frac{P(N(s) = 0, N(t) = 0)}{P(N(t) = 0)} = e^{-\lambda(s-t)}.$$  

since $\{N(s) = 0\} \subseteq \{N(t) = 0\}$. For $s \leq t$, it is clear that the above probability is 1 since $\{N(t) = 0\} \subseteq \{N(s) = 0\}$. The case when $n = 0$ may also be solved by using the memoryless property of exponential distributions.

3. Assume that passengers arrive at a bus station as a Poisson process with rate $\lambda$. Whenever a bus arrives, it picks up all the passengers waiting. Let $W$ be the combined waiting time for all passengers. Compute $E[W]$ in each of the following cases.

(a) The only bus departs after a deterministic time $t$.
(b) One bus departs at time $s$ and another at time $t > s$, where $s, t$ are both deterministic times.
(c) Now assume $T$, the only bus arrival time, is distributed exponentially with parameter $\mu$, independent of the passengers’ arrivals.
(d) Finally, two buses arrive as the first two events in a rate $\mu$ Poisson process.

Solution:

(a) If $S_1, S_2, \ldots$ are the arrival times in $[0, t]$, then the combined waiting time is $W = t - S_1 + t - S_2 + \ldots$. Let $N(t)$ be the number of arrivals in $[0, t]$. We obtain the answer by conditioning on the value of $N(t)$:

$$E[W] = \sum_{k=0}^{\infty} E[W | N(t) = k] \cdot P(N(t) = k)$$

$$= \sum_{k=0}^{\infty} \left( \frac{k}{2} \right) P[N(t) = k]$$

$$= \frac{t}{2} E[N(t)] = \frac{\lambda t^2}{2}.$$

(b) We have two independent Poisson processes in time intervals $(0, s]$ and $(s, t]$, so that

$$E[W] = \frac{\lambda s^2}{2} + \frac{\lambda(t-s)^2}{2}.$$

(c) Suppose $T$ has density $f_T(t)$. Then, using the result obtained in part (a), we get

$$E[W] = \int_0^\infty E[W | T = t] f_T(t) \, dt$$

$$= \int_0^\infty \frac{\lambda t}{2} f_T(t) \, dt = \frac{\lambda}{2} ET^2$$

Since $T \sim exp(\mu)$, we get $E[W] = \frac{\lambda}{\mu^2}$.

(d) Two buses now arrive as first two events in a rate $\mu$ Poisson process. Thus,

$$E[W] = 2 \frac{\lambda}{\mu^2}.$$

4. Let $\{N(t) : t \geq 0\}$ be a Poisson process with rate $\lambda$. Let

$$S_k = \inf\{t \geq 0 : N(t) \geq k\}, \quad k = 1, 2, \ldots$$

be the $k^{th}$ arrival time. Prove that for any $k \geq 1$, we have

$$\lim_{t \to \infty} P(S_k > t) = 0,$$

and hence conclude that $P(S_k < \infty) = 1$. 

Solution: Since \( \{T_k > t\} = \{N(t) \leq k - 1\} \), we have
\[
P(T_k > t) = P(N(t) \leq k - 1) = \sum_{i=0}^{k-1} e^{-\lambda t} \frac{(\lambda t)^i}{i!}.
\]
Taking limit as \( t \to \infty \) on both sides of the above equation, and exchanging the limit and the finite summation on the right hand side yields \( \lim_{t \to \infty} P(T_k > t) = 0 \). Next, we observe that
\[
P(T_k = \infty) = P\left( \bigcap_{n=1}^{\infty} \{T_k > n\} \right) \overset{(a)}{=} \lim_{n \to \infty} P(T_k > n) = 0,
\]
where (a) above follows from continuity of probability measure. Thus, \( P(T_k < \infty) = 1 \).

5. For a Poisson process of rate \( \lambda \), consider the following: let \( X \) be a random variable denoting the number of arrivals in the time interval \( (0, t] \), and \( Y \) be a random variable denoting the number of arrivals in \( (0, t+s] \), where \( t, s \geq 0 \).

(a) Find the conditional pmf of \( Y \) given \( \{X = x\} \), for all \( y \geq 0 \).

(b) Find the joint pmf of \( Y \) and \( X \).

(c) Find the conditional pmf of \( X \) given \( \{Y = y\} \), for all \( x \geq 0 \).

(d) Find \( E[XY] \).

Solution:

(a) By the independent and stationary increments property, we have
\[
P(Y = y | X = x) = P(Y - X = y - x | X = x) = e^{-\lambda t} \frac{(\lambda t)^{y-x}}{(y-x)!}.
\]

(b) We have
\[
P(X = x, Y = y) = P(X = x, Y - X = y - x) = e^{-\lambda t} \frac{(\lambda t)^{x}}{x!} e^{-\lambda s} \frac{(\lambda s)^{y-x}}{(y-x)!}.
\]

(c) We have
\[
P(X = x | Y = y) = \frac{P(X = x, Y = y)}{P(Y = y)} = \frac{e^{-\lambda t} \frac{(\lambda t)^{x}}{x!} e^{-\lambda s} \frac{(\lambda s)^{y-x}}{(y-x)!}}{e^{-\lambda (t+s)} \frac{(\lambda (t+s))^y}{y!}}.
\]
\[
= \left( \frac{t}{t+s} \right)^x \left( 1 - \frac{t}{t+s} \right)^{y-x}.
\]

This, the conditional distribution of \( X \), given \( Y = y \), is binomial, with parameter \( y \) and success probability \( \frac{t}{t+s} \).
(d) We have

\[ E[XY] = E[X(Y - X + X)] \]
\[ = E[X(Y - X)] + E[X^2] \]
\[ = E[X]E[Y - X] + E[X^2] \]
\[ = (\lambda t)(\lambda s) + (\lambda^2 t^2 + \lambda t), \]

where the third line above follows from independence of \( X \) and \( Y - X \) by independent increments property.

6. For a Poisson process of rate \( \lambda \), let random variable \( X(t) \) be the number of arrivals that occur in the time interval \((0, t]\). Let \( T \) be a random variable that is independent of the time instants when arrivals in the Poisson process occur. Suppose that \( T \) has an exponential density with parameter \( \nu \) (mean \( 1/\nu \)). Find distribution of \( X(T) \).

**Solution:** For any \( n \geq 0 \), we have

\[
P(X(T) = n) = \int_{0}^{\infty} P(X(T) = n \mid T = t) f_T(t) \, dt
\]
\[
\stackrel{(a)}{=} \int_{0}^{\infty} P(X(t) = n) f_T(t) \, dt
\]
\[
= \int_{0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \nu e^{-\nu t} \, dt
\]
\[
= \left( \frac{\nu}{\nu + \lambda} \right) \left( \frac{\lambda}{\nu + \lambda} \right)^n
\]
\[
= \left( \frac{\nu}{\nu + \lambda} \right) \left( 1 - \frac{\nu}{\nu + \lambda} \right)^n ,
\]

where \((a)\) above follows from the fact that \( T \) is independent of the process \((X(t) : t \in \mathbb{R}_+)\). Thus, \( X(T) \) follows a geometric distribution on the set \( \{0, 1, 2, \ldots\} \), with parameter \( \frac{\nu}{\nu + \lambda} \).

7. For a Poisson process \( A \) of rate \( \lambda_A \), let \( \tau_1 \) and \( \tau_2 \) be the time instants at which first and second arrivals occur respectively.

(a) Find \( E[\tau_2 \mid \tau_1] \).
(b) Find pdf of \( \tau_1^2 \).
(c) Find the joint pdf of \( \tau_1 \) and \( \tau_2 \).

**Solution:**

(a) We have

\[
E[\tau_2 \mid \tau_1] = E[\tau_1 + (\tau_2 - \tau_1) \mid \tau_1]
\]
\[
= E[\tau_1 \mid \tau_1] + E[\tau_2 - \tau_1 \mid \tau_1]
\]
\[
\stackrel{(a)}{=} \tau_1 + E[\tau_2 - \tau_1]
\]
\[
= \tau_1 + \frac{1}{\lambda_A},
\]

where \((a)\) above follows by noting that the random variable \( \tau_2 - \tau_1 \) is independent of \( \tau_1 \) and...
follows exponential distribution with parameter $\lambda$, a proof of which is as follows:

$$P(\tau_2 - \tau_1 > s | \tau_1) = E[1_{\{\tau_2 - \tau_1 > s\}} | \tau_1]$$
$$= E[1_{\{N(\tau_1+s) - N(\tau_1) = 0\}} | \tau_1]$$
$$\overset{(b)}{=} E[1_{\{N(s) = 0\}}]$$
$$= e^{-\lambda_As} \quad \text{where (b) above follows from the strong independent and stationary increments property.}$$

(b) For any $s > 0$, we have

$$P(\tau_2^2 > s) = P(\tau_1 > \sqrt{s})$$
$$= e^{-\lambda_A\sqrt{s}}$$

from which it follows that $P(\tau_2^2 \leq s) = 1 - e^{-\lambda_A\sqrt{s}}$.

(c) Borrowing from the results on ordered statistics, we note that for any $n \geq 2$, and for any $0 < u < v < t < \infty$, we have

$$f_{\tau_1, \tau_2}(N(t) = n) (u, v) = \frac{n(n-1)}{t^2} \left(1 - \frac{v}{t}\right)^{n-2} \left(1 - \frac{u}{t}\right)^{n-2},$$

from which it follows by the law of total probability that for any $0 < u < v < \infty$, we have

$$f_{\tau_1, \tau_2}(u, v) = \sum_{n=2}^{\infty} f_{\tau_1, \tau_2}(N(t) = n) (u, v) P(N(t) = n)$$
$$= \lambda_A^2 e^{-\lambda_Av} \sum_{n=2}^{\infty} e^{-\lambda_A(t-v)} \frac{(\lambda_A(t-v))^{n-2}}{(n-2)!}$$
$$= \lambda_A^2 e^{-\lambda_Av} \frac{1}{(t-v)^2}.$$