1. Let \((X_n)_{n \geq 0}\) be a two-state Markov chain with state space \(\mathcal{X} = \{0, 1\}\), with the following transition matrix:

\[
\begin{pmatrix}
0 & 1 \\
0 & 1 - p & p \\
1 & q & 1 - q
\end{pmatrix}
\]

Argue that this Markov chain has a unique stationary distribution, and subsequently compute the stationary distribution. Let \(\pi = (\pi_0, \pi_1)\) denote the stationary distribution. Then, we have

\[\pi_0 = (1 - p)\pi_0 + q\pi_1, \quad \pi_0 + \pi_1 = 1,\]

from which it follows that

\[\pi_0 = \frac{q}{p + q}, \quad \pi_1 = \frac{p}{p + q}.
\]

2. Consider a time homogeneous DTMC \((X_n)_{n \geq 0}\) on the state space \(\mathcal{X} = \{0, 1, 2, 3, 4, 5\}\) having the following transition matrix:

\[
\begin{pmatrix}
0 & 1 & 2 & 3 & 4 & 5 \\
0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\
1 & \frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 \\
2 & 0 & 0 & \frac{1}{3} & 0 & \frac{7}{3} \\
3 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & \frac{1}{3} \\
4 & 0 & 0 & \frac{3}{4} & 0 & \frac{1}{4} \\
5 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0
\end{pmatrix}
\]

(a) Determine which states are transient and which are recurrent.

(b) For the above Markov chain, define

\[T_{\{0,1\}} := \inf \{n \geq 1 : X_n \in \{0,1\}\}.
\]

For all \(x \in \{0, \ldots, 5\}\), find

\[P_x(T_{\{0,1\}} < \infty) := P(T_{\{0,1\}} < \infty|X_0 = x).\]

Solution:

(a) Transient states: 3, 5. Recurrent states: 0, 1, 2, 4.

(b) Trivially, \(P_x(T_{\{0,1\}} < \infty) = 1, \quad x \in \{0,1\}\); \(P_x(T_{\{0,1\}} < \infty) = 0, \quad x \in \{2,4\}\). For \(x \in \{3,5\}\), we have

\[P_3(T_{\{0,1\}} < \infty) = P_3\left(\{T_{\{0,1\}} < \infty\} \cap \{T_{\{0,1\}} \leq 1\} \cup \{T_{\{0,1\}} > 1\}\right)\]

\[= P_3\left(\{T_{\{0,1\}} < \infty\} \cap \{T_{\{0,1\}} \leq 1\}\right) + P_3\left(\{T_{\{0,1\}} < \infty\} \cap \{T_{\{0,1\}} > 1\}\right)\]

\[= P(3,0) + P(3,1) + P(3,5)P_5(T_{\{0,1\}} < \infty).
\]

Above, the last equality follows from the fact that

\[P_3\left(\{T_{\{0,1\}} < \infty\} \cap \{T_{\{0,1\}} \leq 1\}\right) = P_3\left(\{T_{\{0,1\}} = 1\}\right) = P(3,0) + P(3,1),\]

and

\[P_3\left(\{T_{\{0,1\}} < \infty\} \cap \{T_{\{0,1\}} > 1\}\right) = P(3,5)P_5(T_{\{0,1\}} < \infty).
\]
Similarly,
\[ P_5(T_{0,1} < \infty) = P(5,1) + P(5,3)P_3 \left( \{ T_{0,1} < \infty \} \right) + P(5,5)P_5 \left( \{ T_{0,1} < \infty \} \right). \]

Substituting \( x = P_3 \left( \{ T_{0,1} < \infty \} \right) \), \( y = P_5(T_{0,1} < \infty) \), we have the following simultaneous equations:
\[
\begin{align*}
x &= \frac{1}{2} + \frac{1}{4} y; \\
y &= \frac{1}{5} + \frac{1}{5} x + \frac{2}{5} y.
\end{align*}
\]

Therefore, \( x = \frac{7}{11}, y = \frac{6}{11} \).

3. Let \( \pi_0 = (\pi_0(x))_{x \in \mathcal{X}} \) and \( \pi_1 = (\pi_1(x))_{x \in \mathcal{X}} \) be two distinct stationary distributions for a Markov chain. Show that for any \( 0 \leq \alpha \leq 1 \), the vector \( \pi_\alpha = (\pi_\alpha(x))_{x \in \mathcal{X}} \) defined by
\[
\pi_\alpha(x) = \alpha \pi_0(x) + (1 - \alpha) \pi_1(x), \quad x \in \mathcal{X},
\]
is also a stationary distribution.

**Solution:** We have
\[
\pi_\alpha P = (\alpha \pi_0 + (1 - \alpha) \pi_1) P
\]
\[
= \alpha (\pi_0 P) + (1 - \alpha) (\pi_1 P)
\]
\[
= \alpha \pi_0 + (1 - \alpha) \pi_1
\]

where \( (a) \) above follows from the fact that \( \pi_0 \) and \( \pi_1 \) are stationary distributions. Hence, \( \pi_\alpha \) is a stationary distribution for every \( \alpha \in [0,1] \).

4. There are \( N \) empty boxes and an infinite collection of balls. At each step \( n \in \mathbb{N} \), a box is chosen at random and a ball is placed in it. Let \( X_n \) be the number of empty boxes after the \( n^{th} \) ball has been placed.

(a) Show that \( (X_n : n \in \mathbb{N}) \) is a Markov chain.

(b) Display the state space and transition probabilities of the above Markov chain.

(c) Classify the states as transient, null recurrent or positive recurrent.

**Solution:**

(a) Consider a historical event \( H_n = \cap_{k=0}^n \{ X_k = x_k \} \) for \( x_1, \ldots, x_k \in \mathcal{X} \). Observe that
\[
P(\{ X_{n+1} = y \} \mid H_n) = \begin{cases} 
\text{probability of picking an empty box,} & \text{if } i = j + 1 \\
\text{probability of picking a box which has one/more balls,} & \text{if } i = j + 1 \\
0, & \text{otherwise}
\end{cases}
\]

Thus, \( P[X_{n+1} = j \mid X_n = i, X_{n-1} = x_{n-1}, \ldots, X_0 = x_0] \) is a function of only \( i \) and \( j \) thereby letting us conclude that \( (X_n) \) is a time homogeneous DTMC.

(b) The transition matrix \( P \) is defined as:
\[
P(i,j) = \begin{cases} 
\frac{N-i}{N}, & \text{if } i = j \\
1 - \frac{N-i}{N}, & \text{if } i = j + 1 \\
0, & \text{otherwise}
\end{cases}
\]

for \( N \geq i \geq 0 \).
5. Let \((X_n)_{n \geq 0}\) be an irreducible, positive recurrent DTMC on \(X = \{0, 1, 2, 3, \ldots\}\). Let \(X_0 = y\). Fix \(x \in X\), \(x \neq y\), and for \(k \geq 1\), define the random variable \(V_k^{(x)}\) as the number of visits to \(x\) between the \((k-1)\)th and \(k\)th return to \(y\). Show that \((V_k^{(x)})_{k \geq 1}\) is an iid sequence.

**Solution:** Define \(T_0 = 0\), and for each \(k \geq 1\), define \(T_k\) to be the \(k\)th return time to \(y\), i.e.,

\[
T_k := \inf\{n > T_{k-1} : X_n = y\}.
\]

Since the DTMC is positive recurrent, it will return back to state \(y\) in finite time with probability 1, i.e., \(P(T_k < \infty) = 1\) for all \(k \geq 1\). This in turn implies that \(P(T_k - T_{k-1} < \infty) = 1\) for all \(k \geq 1\). In what follows, we first demonstrate that the collection \((V_k^{(x)})_{k \geq 1}\) is identically distributed.

Towards this, for any \(n \in \{0, 1, 2, \ldots\}\), we have

\[
P(V_k^{(x)} = n|X_0 = y) = P(V_k^{(x)} = n, T_{k-1} < \infty, T_k - T_{k-1} < \infty|X_0 = y)
\]

\[
= \sum_{m=1}^{\infty} \sum_{l=0}^{\infty} P(V_k^{(x)} = n, T_{k-1} = m, T_k - T_{k-1} = l|X_0 = y)
\]

\[
= \sum_{m,l=1}^{\infty} P(T_{k-1} = m, X_m = y|X_0 = y) \cdot P(V_k^{(x)} = n, X_{m+1} \neq y, \ldots, X_{m+l-1} \neq y, X_{m+l} = y|X_0 = y)
\]

\[
= \sum_{m,l=1}^{\infty} P(T_{k-1} = m, X_m = y|X_0 = y) \cdot P(V_1^{(x)} = n, X_1 \neq y, \ldots, X_l \neq y, X_{l+1} = y|X_0 = y)
\]

\[
= \sum_{m,l=1}^{\infty} P(T_{k-1} = m, X_m = y|X_0 = y) \cdot P(V_1^{(x)} = n, T_1 = l|X_0 = y)
\]

\[
= P(V_1^{(x)} = n|X_0 = y),
\]

where (a) above follows from the strong Markov property and the last line follows by noting that \(P(T_{k-1} < \infty) = 1 \equiv P(T_1 < \infty)\). Thus, the collection \((V_k^{(x)})_{k \geq 1}\) is identically distributed.

We now show that for any \(k \geq 1\), the random variables \(V_k^{(x)}\) and \(V_{k+1}^{(x)}\) are independent. Towards this, it suffices to show that for all \(m, n \in \{0, 1, 2, \ldots\}\), we have

\[
P(V_k^{(x)} = m, V_{k+1}^{(x)} = n|X_0 = y) = P(V_k^{(x)} = m|X_0 = y)P(V_{k+1}^{(x)} = n|X_0 = y).
\]

We have

\[
P(V_k^{(x)} = m, V_{k+1}^{(x)} = n|X_0 = y) = P(V_k^{(x)} = m, V_{k+1}^{(x)} = n, T_{k-1} < \infty, T_k - T_{k-1} < \infty, T_{k+1} - T_k < \infty|X_0 = y)
\]

\[
= \sum_{l,p,q=1}^{\infty} P(V_k^{(x)} = m, V_{k+1}^{(x)} = n, T_{k-1} = l, T_k = p + l, T_{k+1} = p + l + q|X_0 = y).
\]

Using similar arguments as in part (a), by using the strong Markov property, it can be shown that

\[
P(V_k^{(x)} = m, V_{k+1}^{(x)} = n|X_0 = y) = P(V_1^{(x)} = m|X_0 = y)P(V_2^{(x)} = n|X_0 = y)
\]

\[
= P(V_k^{(x)} = m|X_0 = y)P(V_{k+1}^{(x)} = n|X_0 = y),
\]

where the last line above follows from the result in part (a).
6. Consider a time homogeneous DTMC $X = (X_n : n \in \mathbb{N})$ on the state space $\mathcal{X} = \{0, 1, 2, \ldots, N\}$ whose transition probabilities are given by

$$p_{xy} = \begin{cases} 1 - \frac{x}{N}, & y = x + 1, \\ \frac{x}{N}, & y = x - 1. \end{cases}$$

Show that the Markov chain is irreducible and positive recurrent. Further, evaluate $\mu_{xx}$ for all $x \in \mathcal{X}$.

**Solution:** We see that every state is accessible from state $0$. In particular, we see that for every state $x \neq 0$, we have

$$p_{0x} = p_{01}p_{12} \cdots p_{x-1,x} = \prod_{i=1}^{x} \left(1 - \frac{i-1}{N}\right) = \frac{N!}{(N-x)!N^x} > 0.$$  

Similarly, we have

$$p_{x0} = p_{x,x-1}p_{x-1,x-2} \cdots p_{1,0} = \frac{x!}{N^x} > 0.$$  

This shows the irreducibility of the Markov chain. To show the positive recurrence, we would show that there exists a positive invariant distribution for this Markov chain.

Let $\pi$ be the stationary distribution of the DTMC $X$. We can write the global balance equation across cuts as

$$\pi_x p_{x,x+1} = \pi_{x+1} p_{x+1,x}.$$  

The above equation along with the definition of transition probability matrix $P$ gives us

$$\pi_x \frac{N-x}{x+1} = \pi_{x+1}. $$  

Hence, we can write it as

$$\pi_k = \pi_0 \frac{N(N-1)\cdots(N-k+1)}{k!} = \pi_0 \binom{N}{k}. $$  

Since $\sum_k \pi_k = 1$, we get $\pi_k = 2^{-N} \binom{N}{k}$, and hence $\mu_{xx} = 2^N / \binom{N}{x}$.  

7. Consider a finite graph $G = (V,E)$, where $V$ denotes the vertex set and $E$ denotes the edge set. Consider a random walk on the vertices of this graph that at each step involves moving from a vertex to one of its neighbors with equal probability. That is, the transition probabilities given by

$$p_{xy} = \begin{cases} \frac{1}{\text{deg}(x)}, & \text{if vertex } y \text{ is a neighbour of vertex } x, \\ 0, & \text{otherwise}, \end{cases}$$

where $\text{deg}(x)$ denotes the degree of vertex $x \in V$. What is a candidate stationary distribution for this random walk?

**Solution:** Notice that for all $x, y \in V$, we have

$$\text{deg}(x) p_{xy} = \text{deg}(y) p_{yx}.$$  

Summing up the above equation over all $y \in V$, we get

$$\text{deg}(x) \sum_{y \in V} p_{xy} = \sum_{y \in V} \text{deg}(y) p_{yx},$$  

which gives $\text{deg}(x) = \sum_{y \in V} \text{deg}(y) p_{yx}$ for all $x \in V$, since $\sum_{y \in V} p_{xy} = 1$. Thus, a candidate stationary distribution for the given random walk is

$$\pi_x = \frac{\text{deg}(x)}{\sum_{y \in V} \text{deg}(y)}, \quad x \in V.$$
8. Suppose we have two boxes and \(2d\) balls, of which \(d\) are black and \(d\) are red. At start, \(d\) balls are placed in box 1, and the remainder in box 2 (not necessarily as \(d\) black balls in the first box and \(d\) red balls in the second box). In round \(n \in \{1, 2, \ldots\}\), a ball is chosen uniformly at random from each of the boxes, and the two balls are put back in the opposite boxes. Let \(X_0\) denote the number of black balls initially in box 1, and for \(n \geq 1\), let \(X_n\) denote the number of black balls in box 1 after the \(n^{th}\) round.

1. Write down the transition function of the Markov chain \((X_n)_{n \geq 0}\).

2. Find the stationary distribution for the above Markov chain.

**Hint:** You may use the formula
\[
\binom{d}{0}^2 + \cdots + \binom{d}{d}^2 = \left(\frac{2d}{d}\right).
\]

**Solution:**

(a) Let \(Z_{n,i}\) for \(i \in \{1, 2\}\), be an indicator random variable indicating if a black ball was drawn from box \(i\) at time \(n\).

\[
Z_{n,i} = \begin{cases} 
1 & \text{black ball drawn from } i^{th} \text{ box} \\
0 & \text{otherwise} 
\end{cases}
\]

\[
Z_{n,1} = \begin{cases} 
1 & \text{w.p. } X_n/d \\
0 & \text{w.p. } 1 - X_n/d 
\end{cases}
\]

\[
Z_{n,2} = \begin{cases} 
1 & \text{w.p. } 1 - X_n/d \\
0 & \text{w.p. } X_n/d 
\end{cases}
\]

\[
X_n = X_{n-1} - Z_{n-1,1} + Z_{n-1,2}
\]

Clearly, the probability of \(X_n\) differing from \(X_{n-1}\) by strictly more than 1 is 0.

\[P(X_n = x + j|X_{n-1} = x), \text{ for } j \in \{1, -1, 0\}, \text{ is given as follows:}\]

\[P(X_n = x + 1|X_{n-1} = x) = P(Z_{n,2} = 1, Z_{n,1} = 0|X_{n-1} = x) = (1 - x/d)^2;\]

\[P(X_n = x - 1|X_{n-1} = x) = P(Z_{n,2} = 0, Z_{n,1} = 1|X_{n-1} = x) = (x/d)^2;\]

\[P(X_n = x|X_{n-1} = x) = P(Z_{n,2} = 1, Z_{n,1} = 1|X_{n-1} = x) + P(Z_{n,2} = 0, Z_{n,1} = 0|X_{n-1} = x) = 2(x/d)(1 - x/d).\]

(b) For all \(y \in \{0, d\}\),

\[\sum_{x \in \{0, d\}} \pi_x P(x, y) = \pi_y.\]

Specifically for \(y \in \{0, d\}\), we have

\[\pi_1 P(1, 0) = \pi_0, \quad \pi_{d-1} P(d - 1, d) = \pi_d,\]

\[\pi_1 \frac{1}{d^2} = \pi_0, \quad \pi_{d-1} \frac{1}{d^2} = \pi_d.\]
\[ \pi_y = \pi_{y+1}(\frac{y+1}{d})^2 + \pi_y(\frac{y}{d})(1 - \frac{y}{d}) + \pi_{y-1}(1 - \frac{y-1}{d})^2 \]

\[ \pi_y(y/d + 1 - y/d)^2 = \pi_{y+1}(\frac{y+1}{d})^2 + \pi_y(y/d)(1 - \frac{y}{d}) + \pi_{y-1}(1 - \frac{y-1}{d})^2 \]

\[ \pi_y(y/d)^2 + \pi_y(1 - y/d)^2 = \pi_{y+1}(\frac{y+1}{d})^2 + \pi_y(1 - \frac{y-1}{d})^2. \]

Now, since \( \pi_1 = \pi_0 d^2 = \pi_0 (\frac{d}{1})^2 \), we have by substituting \( y = 1 \) in the previous equation, \( \pi_2 = \pi_0 (\frac{d}{2})^2 \). At this point we guess that \( \pi_y = \pi_0 (\frac{d}{y})^2 \) for all \( y \in \{0, d\} \). We will prove this by induction. To that end suppose \( \pi_i = \pi_0 (\frac{d}{y})^2 \), for some \( y \), then we will show that \( \pi_{y+1} = \pi_0 (\frac{d}{y+1})^2 \). Indeed,

\[ \pi_{y+1} = \frac{\pi_y(y)^2 + \pi_y(d - y)^2 - \pi_{y-1}(d - (y - 1))^2}{(y + 1)^2} \]

\[ = \frac{d^2 \pi_0}{(y + 1)^2} \left( \frac{1}{(y-1)!^2(d-y)!^2} + \frac{1}{(y)!^2(d-y-1)!^2} - \frac{1}{(y-1)!^2(d-y)!^2} \right) \]

\[ = \frac{1}{(y+1)!^2(d-y-1)!^2} \pi_0 \]

\[ = (\frac{d}{y+1})^2 \pi_0. \]

Also,

\[ \sum_{y \in \{0,d\}} \pi_y = 1. \]

Since \( \pi_y = \pi_0 (\frac{d}{y})^2 \), \( \forall y \in \{0,d\} \), we have

\[ \pi_0 = \frac{1}{\sum_{y \in \{0,d\}} (\frac{d}{y})^2}. \]

Using the hint,

\[ \pi_0 = \frac{1}{(\frac{2d}{d})}. \]

This results in

\[ \pi_y = \frac{(\frac{d}{y})^2}{(\frac{2d}{d})}, \forall y \in \{0,d\}. \]