1. A random process \( \{X(t) : t \in [0, \infty) \} \) is defined in terms of two random variables \( X_1 \) and \( X_2 \) as follows:
\[
X(t) = X_1 \cos(2\pi f_t t) + X_2 \sin(2\pi f_t t),
\]
where \( f_t \) is a constant. Determine the necessary and sufficient conditions on \( X_1 \) and \( X_2 \) such that \( X(t) \) is wide-sense stationary.

2. Let \( \{X_n : n \in \mathbb{Z}\} \) be a stochastic process defined by
\[
X_n = \begin{cases} 
u_n, & \text{if } n \text{ is odd,} \\ \frac{1}{\sqrt{2}}(\nu_n^2 - 1), & \text{if } n \text{ is even,} \end{cases}
\]
where \( \nu_n \sim \mathcal{N}(0, 1) \). Show that the process \( \{X_n\}_{n \in \mathbb{Z}} \) is wide-sense stationary but not stationary.

3. Let \( \{X_n\}_{n \geq 0} \) be an iid process on the state space \( \mathcal{X} = \{1, \ldots, K\} \), with
\[
P(X_1 = k) = \frac{1}{K}, \quad \forall k \in \{1, \ldots, K\}.
\]
Let \( T_1 = 1 \), and for each \( k \in \{2, 3, \ldots, K\} \), recursively define
\[
T_k := \min\{n > T_{k-1} | X_n \neq X_{T_i} \text{ for all } i = 1, 2, \ldots, k - 1\}.
\]
(Remark: \( T_K \) represents the time taken for the process \( X_n \) to take all the \( K \) values.)
Furthermore, define
\[
S_k := T_k - T_{k-1}, \quad \forall k \in \{2, \cdots K\}.
\]
(a) For all \( k \in \{2, \cdots K\} \), compute \( E[S_k] \).
(b) For all \( k \in \{2, \cdots K\} \), compute \( E[T_k] \). (Hint: Use part (a)).

4. Let \( \{X_n\}_{n \geq 0} \) be a discrete-time Markov chain on a countable state space \( \mathcal{X} \). Prove the following statements:
   (a) For any subsets \( A_0, \ldots, A_{n-1} \subset \mathcal{X} \), and for any \( x, y \in \mathcal{X} \), we have
   \[
P(X_{n+1} = y | X_0 = A_0, \ldots, X_{n-1} \in A_{n-1}, X_n = x) = P(X_{n+1} = y | X_n = x).
   \]
   (b) For \( n < n_1 < n_2 \), and any \( x_0, \ldots, x_{n-1}, x, y_1, y_2 \in \mathcal{X} \), we have
   \[
P(X_{n_1} = y_1, X_{n_2} = y_2 | X_0 = x_0, \ldots, X_{n-1} = x_{n-1}, X_n = x) = P(X_{n_1} = y_1, X_{n_2} = y_2 | X_n = x).
   \]

5. Let \( \{X_n\}_{n \geq 0} \) be a Markov chain on a countable state space \( \mathcal{X} \). Show that for all \( n \geq 1 \) and for all \( x_0, \ldots, x_n \in \mathcal{X} \), the following statement holds:
   \[
P(X_0 = x_0 | X_1 = x_1, \ldots, X_n = x_n) = P(X_0 = x_0 | X_1 = x_1).
   \]

6. Let \( X = \{X_n\}_{n \geq 0} \) be a random process taking values in a discrete set \( \mathcal{X} \). Show that \( X \) is a time homogeneous Markov chain if and only if there exists a function \( f(x,y) \) such that
   \[
P(X_{n+1} = y | X_0 = x_0, X_1 = x_1, \ldots, X_{n-1} = x_{n-1}, X_n = x) = f(x,y)
   \]
holds for all \( x_0, x_1, \ldots, x_{n-1}, x, y \in \mathcal{X} \).

7. Suppose \( \{X_n\}_{n \geq 0} \) is a Markov chain on a countable state space \( \mathcal{X} \), with transition matrix \( P \). Let \( \{Y_n\}_{n \geq 0} \) be another process defined as
   \[
   Y_n = X_{kn}, \quad n \in \{0, 1, 2, \ldots\},
   \]
where \( k \in \{1, 2, \ldots\} \) is a fixed constant. Show that the process \( \{Y_n\}_{n \in \mathbb{N}} \) is also Markov. What is its transition matrix?
8. Let \((X_n)_{n \geq 0}\) be a two-state markov chain with states 0 and 1, with the following transition matrix:
\[
\begin{pmatrix}
0 & 1 \\
(1 - p & p) \\
q & 1 - q
\end{pmatrix}
\]
Assume that \(P(X_0 = 0) = 0.5\). Evaluate the following.

1. \(P(X_1 = 0 | X_0 = 0, X_2 = 0)\).
2. \(P(X_1 \neq X_2)\).
3. For \(p = 0.3\) and \(q = 0.4\), calculate the probability of \(X_{n+4} = 0\) given \(X_n = 0\).

9. Suppose that the numbers of families that check in to a hotel on successive days are independent Poisson random variables with mean \(\lambda > 0\). Also suppose that the number of days that a family stays in the hotel is a geometric random variable with parameter \(p, 0 < p < 1\). (Thus, a family who spent the previous night in the hotel will, independently of how long they have already spent in the hotel, check out the next day with probability \(p\).) Also suppose that all families act independently of each other.

(a) If \(X_n\) denotes the number of families that check in to the hotel at the beginning of day \(n\), then show that \(\{X_n, n \geq 0\}\) is a Markov chain.

(b) Write down the transition probabilities of this Markov chain.

10. We recall the setup of question 4, homework 3. A miner is trapped in a mine containing three doors. The first door leads to a tunnel that takes him to safety after two hours of travel. The second door leads to a tunnel that returns him to the mine after three hours of travel. The third door leads to a tunnel that returns him to the mine after five hours. The miner is, at all times, equally likely to choose any one of the doors. Let \(T\) denote the time taken by the miner to escape the mine.

(a) Come up with a sequence of iid random variables \(X_1, X_2, \ldots\) and a stopping time \(N\) such that \(T\) may be expressed as
\[
T = \sum_{i=1}^{N} X_i.
\]
(Note: You may have to imagine that the miner continues to randomly choose doors even after he reaches safety.)

(b) Use Wald’s lemma to compute \(E[T]\).