1. Let \((X_n)_{n \in \mathbb{N}}\) be a sequence of iid random variables. Assume that \(E[X_j] = \mu\) and \(\text{var}(X_j) = \sigma^2\), where both \(\mu\) and \(\sigma^2\) are finite. Define two new sequences \((M_n)_{n \in \mathbb{N}}\) and \((V_n)_{n \in \mathbb{N}}\) as follows:

\[
M_n := \frac{1}{n} \sum_{j=1}^{n} X_j,
\]

and

\[
V_n := \frac{1}{n} \sum_{j=1}^{n} (X_j - M_n)^2.
\]

(Remark: \(M_n\) and \(V_n\) are known as sample mean and sample variance, respectively.)

Show the following for each \(n \in \mathbb{N}\).

(a) \(E[M_n] = \mu\).

(b) \(\text{var}(M_n) = \frac{\sigma^2}{n}\).

(c) \(E[V_n] = n^{-1} \sigma^2\).

2. Let \(X_n\) be a sequence of independent random variables defined by

\[
P(X_n = 1) = \frac{1}{n}, \quad P(X_n = 0) = 1 - \frac{1}{n}.
\]

Show that \(X_n \overset{m.s.}{\to} 0\) but not almost surely.

(This problem demonstrates that there may be mean-squared convergence but not almost sure convergence.)

3. Let \(\Theta\) be uniformly distributed on \([0, 2\pi]\). For each \(n \in \mathbb{N}\), define two sequences \((X_n)_{n \in \mathbb{N}}\) and \((Y_n)_{n \in \mathbb{N}}\) of random variables as follows:

(a) \(X_n = \cos(n\Theta)\),

(b) \(Y_n = \left(\left|1 - \frac{\Theta}{\pi}\right|\right)^n\).

In each of the above cases, argue in which of the four senses (a.s., m.s., p., d.) do the sequences converge. Identify the limits, if they exist, and justify your answers.

4. Let \((X_n)_{n \in \mathbb{N}}\) be a sequence of iid Bernoulli random variables with \(P(X_n = 0) = P(X_n = 1)\).

(a) Determine if the sequence converges in distribution. If so, what is the distribution of the limit random variable?

(b) Determine if the sequence converges in probability to the above deduced limit random variable.

5. Let \((X_n)_{n \in \mathbb{N}}\) be a sequence of random variables with

\[
P\left(X_n = \frac{1}{2}\right) = \frac{1}{2} = P\left(X_n = -\frac{1}{2}\right).
\]

Show that \(X_n\) converges in mean, in mean-squared sense and almost surely to 0.

6. For a constant \(c \in \mathbb{R}\), show that if \(X_n \overset{d}{\to} c\), then \(X_n \overset{p}{\to} c\).

(Hint: Use the fact that the CDF of the constant random variable \(c\) has a jump at \(c\), with the size of jump equal to 1. Use the fact that the distribution of \(X_n\’s\) converges to the distribution of the constant random variable \(c\) for all points except \(c\).)

7. We are given a sequence \((X_n)_{n \in \mathbb{N}}\) of iid random variables with the following pmf:

\[
P\left(X_n = \frac{1}{2}\right) = \frac{1}{2} = P\left(X_n = -\frac{1}{2}\right).
\]
Prove without using law of large numbers that

\[ M_n := \frac{1}{n} \sum_{j=1}^{n} X_j \]

converges to 0 in probability. (Hint: Use Chebyshev’s inequality on \( M_n \) to get an upper bound for
the probability \( P(|M_n| > \epsilon) \).

8. Let \((X_n)_{n \in \mathbb{N}}\) be a sequence of iid Poisson random variables with parameter \( \lambda = 1 \). For each \( n \in \mathbb{N} \), let \( S_n := \sum_{j=1}^{n} X_j \).

(a) Show that

\[ \lim_{n \to \infty} \frac{S_n - n}{\sqrt{n}} \]

converges in distribution to \( \mathcal{N}(0,1) \).

(b) Also show that \( S_n \) is Poisson random variable with parameter \( n \).

(c) Compute the probability \( P(S_n \leq n) \).

(d) Show that

\[ \lim_{n \to \infty} e^{-n} \left( \sum_{k=0}^{n} \frac{n^k}{k!} \right) = \frac{1}{2}. \]

(Hint: For part (d), use

\[ P(S_n \leq n) = P \left( \frac{S_n - n}{\sqrt{n}} \leq 0 \right) \]

and then argue using the result you proved in part (a).)

9. Let \( f \) be a continuous function on \([0,1]\). Furthermore, let \( \int_0^1 |f(x)| \, dx < \infty \). Define \( I := \int_0^1 f(x) \, dx \).
Let \((U_n)_{n \in \mathbb{N}}\) be a sequence of iid uniform random variables on \([0,1]\). Define

\[ I_n := \frac{1}{n} \sum_{j=1}^{n} f(U_j). \]

Using strong law of large numbers, argue that the sequence \( I_n \) converges to \( I \) almost surely.

(Remark: This is the idea behind Monte-Carlo integration. We sample points uniformly in \([0,1]\) and
then approximate the integral by averaging out the values we get. This is useful when the integral
\( \int_0^1 f(x) \, dx \) cannot be computed in closed form.)

10. Let \((X_n)_{n \in \mathbb{N}}\) be a sequence of random variables, with \( X_n \) being exponentially distributed with mean \( \frac{1}{n} \) for each \( n \in \mathbb{N} \).

(a) Show that the sequence \((X_n)_{n \in \mathbb{N}}\) converges to 0 in mean.

(b) Show that the sequence \((X_n)_{n \in \mathbb{N}}\) converges to 0 in probability.

(c) Show that the sequence \((X_n)_{n \in \mathbb{N}}\) converges to 0 almost surely. (Hint: Use the first part of
Borel-Cantelli lemma.)

(d) Now, suppose that \((X_n)_{n \in \mathbb{N}}\) is a sequence of mutually independent exponential random variables, with mean of \( X_n \) equal to \( \frac{1}{\log(n)} \) for each \( n \in \mathbb{N} \). Check if the statements in part (a), (b) and (c) are still true.