1. Let \((\Omega, \mathcal{F}, P) = ([0, 1], B([0, 1]), \lambda)\), where \(\lambda\) denotes the standard Lebesgue measure on \(\mathbb{R}\).

(a) Let \(X_n = n \cdot 1_{[0, \frac{1}{n}]}\) (here, \(1_A\) denotes the indicator function of the set \(A\)). Sketch the cdf of \(X_n\), and show that \(X_n \xrightarrow{d} 0\).

(b) Show that \(X_n\)’s as defined above are not independent.

2. If \(\sum_{n=1}^{\infty} E[|X_n|^p] < \infty\) for some \(p > 0\), then show that \(X_n \xrightarrow{a.s.} 0\).

3. Let \(X_n\)’s be random variables such that \(P(X_n = 0) = \frac{1}{n} = 1 - P(X_n = 1)\) for all \(n\), and let \(X\) be such that \(P(X = 1) = 1\). Prove that \(X_n\) converges to \(X\) in distribution and in probability (prove both separately. Do not use the fact that convergence in probability implies that in distribution).

4. Let \((\Omega, \mathcal{F}, P) = ([0, 1], B([0, 1]), \lambda)\), where \(\lambda\) denotes the standard Lebesgue measure on \(\mathbb{R}\).

   In each of the cases below, identify the limit and the notion(s) of convergence to this limit.

   (a) \(X_n(\omega) = n^2 \omega \cdot 1_{(0, \frac{1}{n})}(\omega)\)

   (b) \(X_n(\omega) = n \omega - \lfloor n \omega \rfloor\), where \(\lfloor x \rfloor\) denotes the largest integer less than or equal to \(x\)

   (c) \(X_n(\omega) = n \cdot \omega^n\).

5. (Convergence in distribution to convergence in probability) Suppose \(X_n \xrightarrow{d} c\), where \(c \in \mathbb{R}\) is a constant. Then, show that \(X_n \xrightarrow{i.p.} c\).

6. Let \((\Omega, \mathcal{F}, P)\) be a probability space, and let \(X_1, X_2, \ldots\) be a sequence of real-valued random variables defined on \((\Omega, \mathcal{F})\). Let \(A\) be a Borel set such that \(P(X_k \in A) = p\) for all \(k \geq 1\), and let \(Y_1, Y_2, \ldots\) be another sequence of random variables defined as

   \[Y_n := \frac{1}{n} \sum_{k=1}^{n} 1_{\{X_k \in A\}}.\]

   In other words, the random variable \(n \cdot Y_n\) counts the number of times \(X_k \in A\) for \(1 \leq k \leq n\).

   (a) Show that \(Y_n\) converges to \(p\) in probability (do not use weak law of large numbers. Show explicitly using the definition of convergence in probability).

   (b) Does \(Y_n\) converge to \(p\) in the mean-squared sense? Justify your answer.

7. Let \(X_1, X_2, \ldots\) be iid \(\text{Exp}(1)\) random variables. Define \(Y_n := \max\{X_1, \ldots, X_n\}\).
(a) Compute the cdf of $Y_n$.

(b) Let $a, b \in \mathbb{R}$ such that $0 < a < 1 < b$. Show that

$$P(Y_n \leq a \log(n)) \to 0 \text{ as } n \to \infty$$

$$P(Y_n \leq b \log(n)) \to 1 \text{ as } n \to \infty.$$

(c) Deduce that $\frac{Y_n}{\log(n)} \xrightarrow{d} 1$.

8. (*Convergence in distribution need not imply convergence in probability*) Let $X$ be a Ber(0.5) random variable. For each $n \geq 1$, let $Y_n = X$. Let $Y = 1 - X$.

(a) Show that $Y_n \xrightarrow{d} Y$.

(b) Show that $Y_n$ does not converge to $Y$ in probability.