Random Variables - Examples & Exercises

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Aug. 24 2017

Assume \((\Omega, \mathcal{F}, P)\) is a probability space.

1. If \(X : \Omega \to \mathbb{R}\) is a random variable defined with respect to \(\mathcal{F}\), and \(a \in \mathbb{R}\) is any constant, show that \(Y = aX\) is also a random variable with respect to \(\mathcal{F}\).

\(Y = aX\) is a function defined as \(Y(\omega) = aX(\omega), \omega \in \Omega\). Since \(X\) is given to be a random variable, the following statements are equivalent to (21):

\[
\{\omega \in \Omega : X(\omega) \leq y\} \in \mathcal{F} \quad \text{for all } y \in \mathbb{R}, \tag{1}
\]
\[
\{\omega \in \Omega : X(\omega) \geq y\} \in \mathcal{F} \quad \text{for all } y \in \mathbb{R}, \tag{2}
\]
\[
\{\omega \in \Omega : X(\omega) < y\} \in \mathcal{F} \quad \text{for all } y \in \mathbb{R}, \tag{3}
\]
\[
\{\omega \in \Omega : X(\omega) > y\} \in \mathcal{F} \quad \text{for all } y \in \mathbb{R}. \tag{4}
\]

In order to show that \(Y\) is a random variable, it suffices to show that

\[
\{\omega \in \Omega : Y(\omega) \leq x\} = \{\omega \in \Omega : aX(\omega) \leq x\} \in \mathcal{F} \quad \text{for all } x \in \mathbb{R}. \tag{5}
\]

(a) Case 1: Suppose \(a = 0\). Then,

\[
\{\omega \in \Omega : Y(\omega) \leq x\} = \{\omega \in \Omega : aX(\omega) \leq x\}
= \{\omega \in \Omega : 0 \leq x\}
= \begin{cases} \phi, & x < 0 \\ \Omega, & x \geq 0. \end{cases} \tag{6}
\]

From the above description, it is clear that \(\{\omega \in \Omega : Y(\omega) \leq x\} \in \mathcal{F}\) for all \(x \in \mathbb{R}\). Thus, \(Y = aX\) is a random variable when \(a = 0\).

(b) Case 2: Suppose \(a > 0\). Then, for any \(x \in \mathbb{R}\),

\[
\{\omega \in \Omega : Y(\omega) \leq x\} = \{\omega \in \Omega : aX(\omega) \leq x\}
= \{\omega \in \Omega : X(\omega) \leq \frac{x}{a}\} \in \mathcal{F} \tag{7}
\]

since \(\square\) holds with \(y = \frac{x}{a}\). Thus, \(Y = aX\) is a random variable for any \(a > 0\).

(c) Case 3: Suppose \(a < 0\). Then, for any \(x \in \mathbb{R}\),

\[
\{\omega \in \Omega : Y(\omega) \leq x\} = \{\omega \in \Omega : aX(\omega) \leq x\}
= \{\omega \in \Omega : X(\omega) \geq \frac{x}{a}\} \in \mathcal{F} \tag{8}
\]
since (2) holds with \( y = \frac{x}{a} \). Thus, \( Y = aX \) is a random variable for any \( a < 0 \).

2. If \( X \) and \( Y \) are two random variables defined with respect to \( F \), show that \( X + Y \) is also a random variable with respect to \( F \).

Since \( X \) and \( Y \) are given to be random variables, by definition,

\[
\{ \omega \in \Omega : X(\omega) < y \} \in F \quad \text{for all } y \in \mathbb{R}, \tag{9}
\]

\[
\{ \omega \in \Omega : Y(\omega) < y \} \in F \quad \text{for all } y \in \mathbb{R}. \tag{10}
\]

In order to show that \( X + Y \) is a random variable, it suffices to show that

\[
\{ \omega \in \Omega : X(\omega) + Y(\omega) < x \} \in F \quad \text{for all } x \in \mathbb{R}. \tag{11}
\]

Fix an arbitrary \( x \in \mathbb{R} \). Then, \( X(\omega) + Y(\omega) < x \) implies that there exists a rational number \( q \in \mathbb{Q} \) such that \( X(\omega) < q \) and \( Y(\omega) < x - q \). Conversely, if there exists a rational number \( q \in \mathbb{Q} \) such that \( X(\omega) < q \) and \( Y(\omega) < x - q \), then this implies that \( X(\omega) + Y(\omega) < x \). By translating the words “there exists” and “and” into union and intersection of sets respectively, we get that

\[
\{ \omega \in \Omega : X(\omega) + Y(\omega) < x \} = \bigcup_{q \in \mathbb{Q}} \left( \{ \omega \in \Omega : X(\omega) < q \} \cap \{ \omega \in \Omega : Y(\omega) < x - q \} \right) \cap_{\in F \text{ from (9) with } y=q} \cap_{\in F \text{ from (10) with } y=x-q} \in F \text{ since intersection of two events in } F \text{ belongs to } F
\]

belongs to \( F \) since the union over \( z \in \mathbb{Z} \) is a countable union, and countable union of events in \( F \) belongs to \( F \) by the property that \( F \) is a \( \sigma \)-algebra. Thus, \( X + Y \) is a random variable.

**Note 1:** In the above analysis, it is crucial that \( X \) and \( Y \) are both defined with respect to \( F \). In other words, if \( X \) is defined with respect to \( F \) and \( Y \) is defined with respect to a different \( \sigma \)-algebra \( G \), then \( X + Y \) is not a meaningful definition.

**Note 2:** The above problem can also be solved using the fact that a continuous function of random variables is a random variable.

3. If \( X \) and \( Y \) are random variables defined with respect to \( F \), show that \( \max\{X,Y\} \) is also a random variable with respect to \( F \).

Since \( X \) and \( Y \) are given to be random variables, by definition,

\[
\{ \omega \in \Omega : X(\omega) \leq y \} \in F \quad \text{for all } y \in \mathbb{R}, \tag{12}
\]

\[
\{ \omega \in \Omega : Y(\omega) \leq y \} \in F \quad \text{for all } y \in \mathbb{R}. \tag{13}
\]

We need to show that

\[
\{ \omega \in \Omega : \max\{X(\omega),Y(\omega)\} \leq x \} \in F \quad \text{for all } x \in \mathbb{R}. \tag{14}
\]
Fix an arbitrary \( x \in \mathbb{R} \). Then, \( \max\{X(\omega), Y(\omega)\} \leq x \) implies that \( X(\omega) \leq x \) and \( Y(\omega) \leq x \), and the converse is also true. Thus,

\[
\{\omega \in \Omega : \max\{X(\omega), Y(\omega)\} \leq x\} = \{\omega \in \Omega : X(\omega) \leq x\} \cap \{\omega \in \Omega : Y(\omega) \leq x\}
\]

belongs to \( \mathcal{F} \) since intersection of two events in a \( \mathcal{F} \) belongs to \( \mathcal{F} \) by the property that \( \mathcal{F} \) is a σ-algebra. Hence \( \max\{X, Y\} \) is a random variable.

4. Show that if \( X \) is a random variable defined with respect to \( \mathcal{F} \), then \( X^2 \) is also a random variable defined with respect to \( \mathcal{F} \).

Since \( X \) is given to be a random variable, by definition,

\[
\{\omega \in \Omega : X(\omega) \leq y\} \in \mathcal{F} \quad \text{for all} \quad y \in \mathbb{R}, \quad (16)
\]

\[
\{\omega \in \Omega : X(\omega) \geq y\} \in \mathcal{F} \quad \text{for all} \quad y \in \mathbb{R}. \quad (17)
\]

We need to show that

\[
\{\omega \in \Omega : (X(\omega))^2 \leq x\} \in \mathcal{F} \quad \text{for all} \quad x \in \mathbb{R}. \quad (18)
\]

Clearly, since \( (X(\omega))^2 \) is a non-negative real number, \( \{\omega \in \Omega : (X(\omega))^2 \leq x\} = \phi \) for all \( x < 0 \). Fix an arbitrary \( x \geq 0 \). Then,

\[
\{\omega \in \Omega : (X(\omega))^2 \leq x\} = \{\omega \in \Omega : |X(\omega)| \leq \sqrt{x}\} = \{\omega \in \Omega : -\sqrt{x} \leq X(\omega) \leq \sqrt{x}\} = \{\omega \in \Omega : \sqrt{-x} \leq X(\omega)\} \cap \{\omega \in \Omega : X(\omega) \leq \sqrt{x}\}
\]

belongs to \( \mathcal{F} \) since intersection of two events in \( \mathcal{F} \) belongs to \( \mathcal{F} \) by the property that \( \mathcal{F} \) is a σ-algebra. Hence \( X^2 \) is a random variable.

5. Let \( (\Omega, \mathcal{F}) = (\mathbb{R}, \mathcal{B}) \), where \( \mathcal{B} \) denotes the Borel σ-algebra of subsets of \( \mathbb{R} \). If \( B \in \mathcal{B} \), then \( B \) is known as a Borel set. Let \( f : \mathbb{R} \to \mathbb{R} \) be a real-valued function defined on \( \mathbb{R} \). Then, \( f \) is said to be a Borel measurable function if:

\[
f^{-1}(B) \in \mathcal{B} \quad \text{for all} \quad B \in \mathcal{B}, \quad (20)
\]

i.e., if the inverse image (under \( f \)) of every Borel set is a Borel set.

6. If \( X : \Omega \to \mathbb{R} \) is a random variable defined with respect to \( \mathcal{F} \) and \( f : \mathbb{R} \to \mathbb{R} \) is Borel measurable, show that \( f(X) : \Omega \to \mathbb{R} \) is also a random variable with respect to \( \mathcal{F} \).

Since \( X \) is a random variable, by definition,

\[
X^{-1}(A) \in \mathcal{F} \quad \text{for every} \quad A \in \mathcal{B}, \quad (21)
\]
and since $f$ is Borel measurable,

$$f^{-1}(B) \in \mathcal{B} \text{ for all } B \in \mathcal{B}. \quad (22)$$

In order to show that $g = f(X)$ is a random variable, we need to show that

$$g^{-1}(B) \in \mathcal{F} \text{ for every } B \in \mathcal{B}. \quad (23)$$

Fix an arbitrary $B \in \mathcal{B}$. Then,

$$g^{-1}(B) = (f(X))^{-1}(B) = X^{-1}(f^{-1}(B)) = X^{-1}(A) \in \mathcal{F}, \quad (24)$$

where $A = f^{-1}(B) \in \mathcal{B}$ from (22) since $f$ is Borel measurable, and $X^{-1}(A) \in \mathcal{F}$ from (21) since $X$ is a random variable.

**Remark:** Every continuous function is Borel measurable. Hence, if $X : \Omega \to \mathbb{R}$ is a random variable defined with respect to $\mathcal{F}$, and $f : \mathbb{R} \to \mathbb{R}$ is continuous, then $f(X) : \Omega \to \mathbb{R}$ is also a random variable with respect to $\mathcal{F}$. Thus, for example, if $X$ is a random variable, so are $|X|, e^X, X^2, \sin(X), aX + b$ (for any $a, b \in \mathbb{R}$), etc. On similar lines, if $X$ and $Y$ are random variables defined with respect to $\mathcal{F}$, then so are $X + Y, X - Y, \log(|X + Y|)$, etc.

**Miscellaneous exercises:**

Assume $(\Omega, \mathcal{F}, P)$ is a probability space, and all random variables defined below are functions on $\Omega$.

1. If $X$ and $Y$ are random variables defined with respect to $\mathcal{F}$, show that the following are also random variables defined with respect to $\mathcal{F}$ (do not use the fact that continuous functions of random variables are random variables):

   (i) $|X|, |Y|
   (ii) $X - Y$
   (iii) $XY$
   (iv) $\min\{X, Y\}$
   (v) $X_+ := \max\{X, 0\}, X_- := -\min\{X, 0\}$
   (vi) $|X - Y|$.

2. If $X$ and $Y$ are random variables defined with respect to $\mathcal{F}$, show that

   $$\{\omega \in \Omega : X(\omega) = Y(\omega)\} \in \mathcal{F}.$$ 

3. Let $X : \Omega \to \mathbb{R} \cup \{-\infty, +\infty\}$ be a random variable defined with respect to $\mathcal{F}$. Then, show that $\{\omega \in \Omega : |X(\omega)| = \infty\} \in \mathcal{F}$ (in this example, $X$ is allowed to take the values $-\infty$ and $+\infty$).
4. Prove, by induction, that for any \( n \geq 1 \), if \( X_1, \ldots, X_n \) are random variables, all defined with respect to \( \mathcal{F} \), then the following are also random variables with respect to \( \mathcal{F} \):

(a) \( \frac{X_1 + \ldots + X_n}{n} \)

(b) \( \frac{X_1 + \ldots + X_n}{\sqrt{n}} \).

5. Let \( X \) be a random variable defined with respect to \( \mathcal{F} \), and suppose \( X_1, X_2, \ldots \) is a sequence of random variables, all defined with respect to \( \mathcal{F} \). Then, show that for any \( \epsilon > 0 \),

\[
\{ \omega \in \Omega : |X_n(\omega) - X(\omega)| > \epsilon \} \in \mathcal{F}
\]

for all \( n \geq 1 \).