Why timely update?

- Critical to know the status update before decision making
Potential Scenarios

- Cyber-physical systems: Environmental/health monitoring
- Internet of Things: Real-time actuation/control
Link Model

Context

- **Point-to-point communication** with limited to no feedback
- Reliability through finite block-length **coding**
Source state $X(t)$ can be represented by $m$ bits

State difference between $n$ realizations can be represented by $k < m$ bits
Problem Statement

Question
How to encode message at the temporally correlated source for timely update? Should one send the current state or the difference between the current and the past state?

Answer
It depends on the feedback
Coding Model

- Finite length code of $n$ bits with permutation invariant code

Updates

- True Update: current state $X(t)$ of $m$ bits is encoded to $n$ bit codeword $X^n$
- Incremental Update: the state difference $X(t) - X(t - n)$ of $k$ bits encoded to $n$ bit codeword $X^n$
Channel Model

- Each transmitted bit of the codeword $X^n$ erased iid with probability $\epsilon$

Erasure Distribution
Number of erasures is Binomial with parameter $(n, \epsilon)$
Decoding and Reception

Receiver Timing
Reception at time $t + n$ of $n$ bits sent at time $t$ after $n$ channel uses

Probability of Decoding Failure

- True updates: $p_1 = EP(n, n - m, E)$
- Incremental updates: $p_2 = EP(n, n - k, E)$
- Monotonicity: $0 < p_2 < p_1 < 1$
Performance Metric

- Last successfully decoded source state at time $t$ was generated at $U(t)$
- *Information age* $A(t)$ at time $t$ as
  \[ A(t) = t - U(t). \]
- Limiting value of average age
  \[ \lim_{t \in \mathbb{R}} \frac{1}{t} \sum_{s=1}^{t} A(s). \]
Update Transmission Schemes

True Updates

- Each opportunity send *true update*

Incremental Updates without Feedback

- Periodically send the *true update* after $q$ updates
- In between true updates, send *incremental updates*.

Incremental Updates with Feedback

- Send the *true update* after each decoding failure
- In between true updates, send *incremental updates*. 
Renewal Reward Theorem: I

- Let \( N(t) \) be a counting process with event instants \( S_i \) for \( i \)th event
- Inter-renewal times \( T_i = S_i - S_{i-1} \) are iid
- Reward in \( i \)th renewal interval \( R_i \) is iid

Renewal Reward Theorem
For cumulative reward process \( R(t) = \sum_{i=1}^{N(t)} R_i \), the limiting average reward

\[
\mathbb{E}R = \lim_{t \in \mathbb{R}} \frac{R(t)}{t} = \frac{\mathbb{E}R_i}{\mathbb{E}T_i}
\]
 Renewal Reward Theorem: II

- Time instant $S_i$ of the $i$th successful reception of the true update
- For all three schemes, the $i$th inter-renewal time $T_i = S_i - S_{i-1}$ is iid
- Associated counting process $N(t)$ is a renewal process
Accumulated age in $i$th renewal period

\[ S(T_i) = \sum_{t=S_{i-1}}^{S_i-1} A(t) \]

is also iid

By renewal reward theorem, the limiting average age is

\[ \mathbb{E}A \triangleq \lim_{t \to \infty} \frac{1}{t} \sum_{s=1}^{t} A(s) = \mathbb{E}S(T_i)/\mathbb{E}T_i. \]
Age Sample Path

- Inter-renewal time $T_i = nZ_i$
- Number of true update in $i$th renewal interval $Z_i$
- $\{Z_i : i \in \mathbb{N}\}$ is iid geometric with success parameter $(1 - p_1)$
Mean Age

Theorem
Limiting average age for the true update scheme is a.s.

\[ \mathbb{E} A \triangleq \lim_{t \to \infty} \frac{1}{t} \sum_{s=1}^{t} A(s) = \frac{n-1}{2} + \frac{n}{1 - p_1}. \]

Proof.
Cumulative age for \( i \)th renewal interval is

\[ S(nZ_i) = \sum_{j=0}^{nZ_i-1} (n+j) = n^2 Z_i + nZ_i(nZ_i - 1)/2. \]
Age Sample Path

- Inter-renewal time $T_i = nqZ_i$
- Number of successfully decoded contiguous incremental updates $W_i - 1$ in the $i$th interval
Incremental Update without Feedback

**Lemma**
For each renewal interval, the number of successful receptions $1 \leq \bar{W}_i \leq q$, and is independent of the number of true updates $Z_i$. Further, the sequence $\{\bar{W}_i : i \in \mathbb{N}\}$ is iid and distributed as truncated geometric

$$
\Pr\{\bar{W}_i = k\} = \begin{cases} 
(1 - p_2)^{k-1}p_2, & 1 \leq k < q, \\
(1 - p_2)^{q-1}, & k = q.
\end{cases}
$$
Mean Age

Theorem

Limiting average age for the incremental updates without feedback is

\[ \mathbb{E} \bar{A} \triangleq \lim_{t \to \infty} \frac{1}{t} \sum_{s=1}^{t} \bar{A}(s) = \frac{\mathbb{E} T_i^2}{2 \mathbb{E} T_i} + \frac{n^2 \mathbb{E} \bar{W}_i (\bar{W}_i - 1)}{2 \mathbb{E} T_i} \]

\[ - \left( n \mathbb{E} (\bar{W}_i - 2) + \frac{1}{2} \right). \]

Proof.

Cumulative age \( S(T_i) \) in the \( i \)th renewal interval is

\[ S(T_i) = \sum_{j=1}^{\bar{W}_i-1} \sum_{k=0}^{n-1} (n + k) + \sum_{j=n(\bar{W}_i-1)}^{T_i-1} (n + j - n(\bar{W}_i - 1)), \]

\[ = \frac{n^2 \bar{W}_i (\bar{W}_i - 1)}{2} + \frac{T_i^2}{2} - \left( n(\bar{W}_i - 2) + \frac{1}{2} \right) T_i. \]
Age Sample Path

- Inter-renewal time $T_i = nZ_i + nW_i$
- Number of incremental updates $W_i$ in $i$th renewal interval
- $\{W_i : i \in \mathbb{N}\}$ are iid geometric with success parameter $p_2$
Mean Age

Theorem

Limiting average age for the incremental updates with feedback is

$$
\mathbb{E}\hat{A} \triangleq \lim_{t \to \infty} \frac{1}{t} \sum_{s=1}^{t} \hat{A}(s) = \frac{3n - 1}{2} + \frac{n(\mathbb{E}Z_i^2 + \mathbb{E}Z_i)}{2(\mathbb{E}W_i + \mathbb{E}Z_i)}.
$$

Proof.

Cumulative age $S(T_i)$ over the $i$th renewal period $T_i$ is

$$
S(T_i) = \sum_{j=1}^{W_i-1} \sum_{k=0}^{n-1} (n + k) + \sum_{k=0}^{T_i-n(W_i-1)-1} (n + k)
= (3n - 1) T_i/2 + n^2 (Z_i + 1) Z_i/2.
$$
Theorem
The mean age for the three schemes satisfy,

\[ E\hat{A} \leq EA \leq E\bar{A}. \]
Random Coding Scheme

- Probability of decoding failure is

\[ P(n, n - r, E) = 1 - \prod_{i=0}^{E-1} \left(1 - 2^i(n-r)\right). \]

- Number of erasure \( E \) has binomial distribution with parameter \((n, \epsilon)\)

- For true update \( p_1 = \mathbb{E}P(n, n - m, E) \) and incremental update \( p_2 = \mathbb{E}P(n, n - k, E) \).
\( n = 120, \ k = 10, \ \epsilon = 0.001 \) and \( q \in \{2, 6\} \)
\( n = 120, \ m = 105, \ \epsilon = 0.1 \) and \( q \in \{2, 6\} \)
\( n = 120, \ m = 105 \) and \( q \in \{2, 6\} \)
Discussion and Concluding Remarks

Main Contributions

- Integration of coding and renewal techniques to study timely communication for delay-sensitive traffic
- We model channel unreliability by the erasure channel
- Incremental updates only when there is feedback availability

Avenues of Future Research

- Extend results to structured sources
- Extend results to correlated finite-state erasure and error channels
- Impact of other coding schemes on timeliness
Thank You