Lecture-28: Poisson processes on the half-line

1 Equivalent characterizations

A counting process $N$ has the **completely independence property**, if for any collection of finite disjoint and bounded sets $A_1, \ldots, A_k \in \mathcal{B}$,

$$P \left( \bigcap_{i=1}^{k} \{ N(A_i) = n_i \} \right) = \prod_{i=1}^{k} P\{N(A_i)=n_i\}.$$  

**Theorem 1.1.** Distribution of a simple point process is completely determined by void probabilities.

**Proof.** We will show this by induction on the number of points in a bounded set $A \in \mathcal{B}$. We assume that $(A_k \in \mathcal{B} : k \in \mathbb{N})$ is a sequence of sufficiently small sets partitioning $A$ such that $N(A_i) \in \{0,1\}$ for all $i \in \mathbb{N}$. We assume that $N(A) = n$ and by the induction hypothesis $P\{N(B) = n-1\}$ can be completely characterized by the void probabilities for all bounded sets $B \in \mathcal{B}$. Then, we can write

$$P\{N(A) = n\} = \sum_{k \in \mathbb{N}} P\{N(A_k) = 1, N(A \setminus A_k) = n-1\} = \sum_{k \in \mathbb{N}} (P\{N(A \setminus A_k) = n-1\} - P\{N(A) = n-1\}).$$


**Theorem 1.2 (Equivalences).** Following are equivalent for a simple counting process $N = (N(A) : A \in \mathcal{B})$.

1. Process $N$ is Poisson with locally finite intensity measure $\Lambda$.
2. For each bounded $A \in \mathcal{B}$, we have $P\{N(A) = 0\} = e^{-\Lambda(A)}$.
3. For each bounded $A \in \mathcal{B}$, the number of points $N(A)$ is a Poisson with parameter $\Lambda(A)$.
4. Process $N$ has the completely independence property, and $\mathbb{E}N(A) = \Lambda(A)$.

**Proof.** We will show that $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 1$.

$1 \Rightarrow 2$. It follows from the definition of Poisson point processes and definition of Poisson random variables.

$2 \Rightarrow 3$. From Theorem 1.1, we know that void probabilities determine the entire distribution.

$3 \Rightarrow 4$. We will show this in two steps.

**Mean:** Since the distribution of random variable $N(A)$ is Poisson, it has mean $\mathbb{E}N(A) = \Lambda(A)$.

**CIP:** For disjoint and bounded $A_1, \ldots, A_k \in \mathcal{B}$ and $A = \bigcup_{i=1}^{k} A_i$, we have $N(A) = N(A_1) + \cdots + N(A_k)$. Taking expectations on both sides, we get $\Lambda(A) = \Lambda(A_1) + \cdots + \Lambda(A_k)$. From the number of partitions $n_1 + \cdots + n_k = n$, we can write

$$P\{N(A) = n\} = \frac{1}{n!} \sum_{n_1+\cdots+n_k=n} \binom{n}{n_1, \ldots, n_k} P\{N(A_1) = n_1, \ldots, N(A_k) = n_k\}.$$  

Using the definition of Poisson distribution, we can write the left hand side of the above equation as

$$P\{N(A) = n\} = e^{-\Lambda(A_i)} \frac{\Lambda(A_i)^n}{n!} = \prod_{i=1}^{k} e^{-\Lambda(A_i)} \frac{(\sum_{i=1}^{k} \Lambda(A_i))^n}{n!} = \frac{1}{n!} \sum_{n_1+\cdots+n_k=n} \binom{n}{n_1, \ldots, n_k} \prod_{i=1}^{k} e^{-\Lambda(A_i)} \Lambda(A_i)^{n_i}.$$
Equating each term in the summation, we get

\[ P \{ N(A_1) = n_1, \ldots, N(A_k) = n_k \} = \prod_{i=1}^{k} P \{ N(A_i) = n_i \}. \]

iv \implies i_v. Due to complete independence property and since void probabilities describe the entire distribution, it suffices to show that \( P \{ \Phi(A) = 0 \} = e^{-\Lambda(A)} \) for all bounded \( A \in \mathcal{B} \). For disjoint and bounded \( A_1, \ldots, A_k \in \mathcal{B} \) and \( A = \bigcup_{i=1}^{k} A_i \), we have

\[ \Lambda(A) = \sum_{i=1}^{k} \Lambda(A_i), \quad \text{and} \quad -\ln P \{ \Phi(A) = 0 \} = -\sum_{i=1}^{k} \ln P \{ \Phi(A_i) = 0 \}. \]

This implies that \( -\ln P \{ \Phi(A) = 0 \} = \Lambda(A) \), and the result follows.

\[ \square \]

## 2 Simple point processes on the half-line

A stochastic process defined on the half-line \( (N(t) : t \geq 0) \) is a **counting process** if

1. \( N(0) = 0 \), and
2. for each \( \omega \in \Omega \), the map \( t \mapsto N(t) \) is non-decreasing, integer valued, and right continuous.

Each discontinuity of the sample path of the counting process can be thought of as a jump of the process, as shown in Figure 1. A simple counting process has the unit jump size almost surely. General point processes in higher dimension don’t have any inter-arrival time interpretation.

![Figure 1: Sample path of a simple counting process.](image)

**Lemma 2.1.** A counting process has finitely many jumps in a finite interval \((0, t]\).

The points of discontinuity are also called the arrival instants of the point process \( N(t) \). The **nth arrival instant** is a random variable denoted \( S_n \), such that

\[ S_0 = 0, \quad S_n = \inf \{ t \geq 0 : N(t) \geq n \}, \quad n \in \mathbb{N}. \]

The **inter arrival time** between \((n-1)\)th and \(n\)th arrival is denoted by \( X_n \) and written as \( X_n = S_n - S_{n-1} \). For a simple point process, we have

\[ P \{ X_n = 0 \} = P \{ X_n \leq 0 \} = 0. \]

**Lemma 2.2.** Simple counting process \((N(t), t \geq 0)\) and arrival process \((S_n : n \in \mathbb{N})\) are inverse processes, i.e.

\[ \{ S_n \leq t \} = \{ N(t) \geq n \}. \]
Proof. Let $\omega \in \{S_n \leq t\}$, then $N(S_n) = n$ by definition. Since $N$ is a non-decreasing process, we have $N(t) \geq N(S_n) = n$. Conversely, let $\omega \in \{N(t) \geq n\}$, then it follows from definition that $S_n \leq t$.

**Corollary 2.3.** The following identity is true.

\[ \{S_n \leq t, S_{n+1} > t\} = \{N(t) = n\}. \]

**Proof.** It is easy to see that $\{S_{n+1} > t\} = \{S_n \leq t\}^c = \{N(t) \geq n + 1\}^c = \{N(t) < n + 1\}$. Hence,

\[ \{N(t) = n\} = \{N(t) \geq n, N(t) < n + 1\} = \{S_n \leq t, S_{n+1} > t\}. \]

**Lemma 2.4.** Let $F_n(x)$ be the distribution function for $S_n$, then $P_n(t) \equiv P\{N(t) = n\} = F_n(t) - F_{n+1}(t)$.

**Proof.** It suffices to observe that following is a union of disjoint events,

\[ \{S_n \leq t\} = \{S_n \leq t, S_{n+1} > t\} \cup \{S_n \leq t, S_{n+1} \leq t\}. \]

**Corollary 2.5 (Poisson process on the half-line).** A collection $(N(t) : t \in \mathbb{R}_+)$ of random variable indexed by time $t$ is a Poisson process with intensity measure $\Lambda$ iff

(a) Starting with $N(0) = 0$, the process $N(t)$ takes a non-negative integer value for all $t \in \mathbb{R}_+$;
(b) the increment $N(t + s) - N(t)$ is surely nonnegative for any $s \in \mathbb{R}_+$;
(c) the increments $N(t_1), N(t_2) - N(t_1), \ldots, N(t_n) - N(t_{n-1})$ are independent for any $0 < t_1 < t_2 < \cdots < t_{n-1} < t_n$;
(d) the increment $N(t + s) - N(t)$ is distributed as Poisson random variable with parameter $\Lambda((t, t + s])$.

The Poisson process is homogeneous with intensity $\lambda$, iff in addition to conditions (a), (b), (c), the distribution of the increment $N(t + s) - N(t)$ depends on the value $s \in \mathbb{R}_+$ but is independent of $t \in \mathbb{R}_+$. That is, the increments are stationary.

**Proof.** We have already seen that definition of Poisson processes implies all four conditions. Conditions (a) and (b) imply that $N$ is a simple counting process on the half-line, condition (c) is the complete independence property of the point process, and condition (d) provides the intensity measure. The result follows from the equivalence iv. in Theorem 1.2.

**Proposition 2.6.** A simple Poisson point process on half-line is Markov.

**Proof.** Let $\mathcal{F}_t = \sigma(N(s) : s \leq t)$ be the history of the process until time $t$. Then, from the independent increment property of Poisson processes, we have for any historical event $H_s \in \mathcal{F}_s$

\[
P\{N(t) = n \mid H_s \cap \{N(s) = k\}\} = P\{N(t) - N(s) = n - k \mid H_s \cap \{N(s) = k\}\} = P\{N(t) = n \mid \{N(s) = k\}\}.
\]

For a homogeneous Poisson point process, the process is homogeneously Markov with $P\{N(t) = n \mid \{N(s) = k\}\} = P\{N(t - s) = n - k\} = e^{-\lambda(t-s)} \frac{\lambda(t-s)^{n-k}}{(n-k)!}$.

**Theorem 2.7.** A simple point process on half-line is strongly Markov.
3 IID exponential inter-arrival times characterization

Proposition 3.1. A simple counting process \((N(t) : t \geq 0)\) is a homogeneous Poisson process with a finite positive rate \(\lambda\), iff the inter-arrival times \((X_n : n \in \mathbb{N})\) are iid random variables with an exponential distribution of rate \(\lambda\).

Proof. We first assume the iid exponentially distributed inter-arrival times to show that for the simple counting process \(N\) and each positive real \(t \in \mathbb{R}_+\), the random variable \(N(t)\) is Poisson with parameter \(\lambda t\), and hence \(N\) is homogeneous Poisson with rate \(\lambda\) from the equivalence \(i\) in Theorem \[1.2\].

For the converse, let \(N(t)\) be a simple homogeneous Poisson point process on half-line with rate \(\lambda\). From equivalence \(iii\) in Theorem \[1.2\] we obtain for any positive integer \(t\),

\[ P\{X_1 > t\} = P\{N(t) = 0\} = e^{-\lambda t}. \]

It suffices to show that inter-arrivals time sequence \((X_n : n \in \mathbb{N})\) is iid. We can show that \(N\) is Markov process with strong Markov property. Since the sequence of ordered points \((S_n : n \in \mathbb{N})\) is a sequence of stopping times for the counting process, it follows from the strong Markov property of this process that \((N(S_n + t) - N(S_n) : t \geq 0)\) is independent of \(S_n\) and \(N(S_n)\). Further, we see that

\[ X_{n+1} = \inf\{t > 0 : N(S_n + t) - N(S_n) = 1\}. \]

It follows that \((X_n : n \in \mathbb{N})\) is an independent sequence. For homogeneous Poisson point process, we have \(N(S_n + t) - N(S_n) = N(t)\) in distribution, and hence \(X_{n+1}\) has same distribution as \(X_1\) for each \(n \in \mathbb{N}\). \(\square\)

For many proofs regarding Poisson processes, we partition the sample space with the disjoint events \(\{N(t) = n\}\) for \(n \in \mathbb{N}_0\). We need the following lemma that enables us to do that.

Lemma 3.2. For any finite time \(t > 0\), a Poisson process is finite almost surely.

Proof. By strong law of large numbers, we have

\[ \lim_{n \to \infty} \frac{S_n}{n} = E[X_1] = \frac{1}{\lambda} \quad \text{a.s.} \]

Fix \(t > 0\) and we define a sample space subset \(M = \{\omega \in \Omega : N(\omega, t) = \infty\}\). For any \(\omega \in M\), we have \(S_n(\omega) \leq t\) for all \(n \in \mathbb{N}\). This implies \(\limsup_n \frac{S_n}{n} = 0\) and \(\omega \notin \{\lim_n \frac{S_n}{n} = \frac{1}{\lambda}\}\). Hence, the probability measure for set \(M\) is zero. \(\square\)

3.1 Distribution functions

Lemma 3.3. Moment generating function of arrival times \(S_n\) is

\[ M_{S_n}(t) = \mathbb{E}[e^{tS_n}] = \begin{cases} \lambda^n (t/\lambda)^{-n}, & t < \lambda \\ \infty, & t \geq \lambda. \end{cases} \]

Lemma 3.4. Distribution function of \(S_n\) is given by \(F_n(t) = P\{S_n \leq t\} = 1 - e^{-\lambda t} \sum_{k=0}^{n-1} (\lambda t)^k / k!\).

Theorem 3.5. Density function of \(S_n\) is Gamma distributed with parameters \(n\) and \(\lambda\). That is,

\[ f_n(s) = \frac{\lambda (\lambda s)^{n-1}}{(n-1)!} e^{-\lambda s}. \]

Theorem 3.6. For each \(t > 0\), the distribution of Poisson process \(N(t)\) with parameter \(\lambda\) is given by

\[ P\{N(t) = n\} = e^{-\lambda t} \frac{(\lambda t)^n}{n!}. \]

Further, \(\mathbb{E}[N(t)] = \lambda t\), explaining the rate parameter \(\lambda\) for Poisson process.

Proof. Result follows from density of \(S_n\) and recognizing that \(P_n(t) = F_n(t) - F_{n+1}(t)\). \(\square\)
Corollary 3.7. Distribution of arrival times $S_n$ is

$$F_n(t) = \sum_{j \geq n} P_j(t), \quad \sum_{n \in \mathbb{N}} F_n(t) = \mathbb{E}N(t).$$

Proof. First result follows from the telescopic sum and the second from the following observation.

$$\sum_{n \in \mathbb{N}} F_n(t) = \mathbb{E} \sum_{n \in \mathbb{N}} 1 \{ N(t) \geq n \} = \sum_{n \in \mathbb{N}} P \{ N(t) \geq n \} = \mathbb{E}N(t).$$

A Poisson process is not a stationary process. That is, the finite dimensional distributions are not shift invariant. This is clear from looking at the first moment $\mathbb{E}N(t) = \lambda t$, which is linearly increasing in time.