Lecture-26: Poisson Processes

1 Poisson and exponential random variables

A non-negative integer valued random variable $N \in \mathbb{N}_0$ is called Poisson if for some constant $\lambda > 0$, we have

$$P\{N = n\} = e^{-\lambda} \frac{\lambda^n}{n!}.$$ 

It is easy to check that $\mathbb{E}N = \text{Var}N = \lambda$. Furthermore, the moment generating function $M_N(t) = \mathbb{E}e^{tN} = e^{\lambda(e^t - 1)}$ exists for all $t \in \mathbb{R}$.

1.1 Memoryless distribution

A random variable $X$ with continuous support on $\mathbb{R}_+$, is called memoryless if for all positive reals $t, s \in \mathbb{R}_+$, we have

$$P\{X > s\} = P(\{X > t + s\} | \{X > t\}).$$

Proposition 1.1. The unique memoryless distribution function with continuous support on $\mathbb{R}_+$ is the exponential distribution.

Proof. Let $X$ be a random variable with a memoryless distribution function $F : \mathbb{R}_+ \to [0,1]$. It follows that $\bar{F}(t) \triangleq 1 - F(t)$ satisfies the semi-group property

$$\bar{F}(t + s) = \bar{F}(t)\bar{F}(s).$$

Since $\bar{F}(x) = P\{X > x\}$ is non-increasing in $x \in \mathbb{R}_+$, we have $\bar{F}(x) = e^{\theta x}$, for some $\theta < 0$ from Lemma A.1.

2 Simple point processes

Consider the $d$-dimensional Euclidean space $\mathbb{R}^d$, and the collection of Borel measurable subsets $B(\mathbb{R}^d)$ of the above Euclidean space. A simple point process is a random countable collection of distinct points $\Phi = \{S_n \in \mathbb{R}^d: n \in \mathbb{N}\}$, such that $|S_n| \to \infty$ as $n \to \infty$.

Example 2.1 (Simple point process on the half-line). We can simplify this definition for $d = 1$. In $\mathbb{R}_+$, one can order the points $(S_n : n \in \mathbb{N})$ of the point process $\Phi$, such that $S_1 < S_2 < \cdots < S_n < \cdots$, and $\lim_{n \in \mathbb{N}} S_n = \infty$. The Borel measurable sets for $\mathbb{R}_+$ are generated by the collection of half-open intervals $\{(0, t) : t \in \mathbb{R}_+\}$.

Point processes can model many interesting physical processes.

1. Arrivals at classrooms, banks, hospital, supermarket, traffic intersections, airports etc.
2. Location of nodes in a network, such as cellular networks, sensor networks, etc.
Corresponding to a point process $\Phi$, we denote the number of points in a set $A \in \mathcal{B}(\mathbb{R}^d)$ by $N(A) = \sum_{n \in \mathbb{N}} 1_{\{S_n \in A\}}$, where we have $N(\emptyset) = 0$. Then $N = (N(A) : A \in \mathcal{B}(\mathbb{R}^d))$ is called a counting process for the point process $\Phi$. A counting process is simple if the underlying process is simple.

**Remark 1.** Let $N \in \mathcal{B}(\mathcal{X})$ be the counting process for the point process $\Phi \in \mathcal{X}$.

i. Note that the point process $\Phi$ and the counting process $N$ carry the same information.

ii. The distribution of point process $\Phi$ is completely characterized by the finite dimensional distributions $(N(A_1), \ldots, N(A_k) : \text{bounded } A_1, \ldots, A_k \in \mathcal{B})$ for some finite $k \in \mathbb{N}$.

**Example 2.2 (Simple point process on the half-line).** The number of points in the half-open interval $(0, t]$ is denoted by $N(t) \triangleq N((0, t]) = \sum_{n \in \mathbb{N}} 1_{\{S_n \in (0, t]\}}$. Since the Borel measurable sets $\mathcal{B}(\mathbb{R}_+)$ are generated by half-open intervals $\{0, t] : t \in \mathbb{R}_+\}$, we denote the counting process by $(N(t) : t \in \mathbb{R}_+)$. For $s < t$, the number of points in interval $(s, t]$ is $N((s, t]) = N((0, t]) - N((0, s]) = N(t) - N(s)$.

For any $k \in \mathbb{Z}_+$ and $n_i \in \mathbb{N}_0$ for $i \in [k]$, the Poisson point process of intensity measure $\Lambda$ is defined by its finite dimensional distribution

$$P\{N(A_1) = n_1, \ldots, N(A_k) = n_k\} = \prod_{i=1}^{k} \left( e^{-\Lambda(A_i)} \frac{\Lambda(A_i)^{n_i}}{n_i!} \right),$$

for all bounded mutually disjoint sets $A_1, \ldots, A_k \in \mathcal{B}$. If $\Lambda(A) = \lambda |A|$, then we call $\Phi$ a homogeneous Poisson point process and $\lambda$ is its intensity.

**Theorem 2.3.** Following are equivalent for a simple counting process $N$.

i. Process $N$ is Poisson with intensity measure $\Lambda$.

ii. Process $N$ has the completely independence property, and $\mathbb{E}N(A) = \Lambda(A)$. A counting process $N$ has the completely independence property, if for any collection of finite disjoint and bounded sets $A_1, \ldots, A_k \in \mathcal{B}$,

$$P \bigcap_{i=1}^{k} \{N(A_i) = n_i\} = \prod_{i=1}^{k} P\{N(A_i) = n_i\}.$$

iii. The intensity measure $\Lambda$ is bounded for bounded $A \in \mathcal{B}$, and $N(A)$ is a Poisson with parameter $\Lambda(A)$.

**Remark 2.** $\Phi$ is a Poisson point process iff for all $k \in \mathbb{Z}_+$ and all bounded, mutually disjoint $A_1, \ldots, A_k \in \mathcal{B}$, the random vector $(N(A_1), \ldots, N(A_k))$ is a vector of independent Poisson random variables with parameters $\Lambda(A_1), \ldots, \Lambda(A_k)$ respectively. In particular, we have $\mathbb{E}N(A) = \Lambda(A)$ for all subsets $A \in \mathcal{B}$.

**Theorem 2.4 (Poisson process on the half-line).** A collection $(N(t) : t \in \mathbb{R}_+)$ of random variable indexed by time $t$ is a Poisson process iff

(a) Starting with $N(0) = 0$, the process $N(t)$ takes a non-negative integer value for all $t \in \mathbb{R}_+$;

(b) the increment $N(t + s) - N(t)$ is surely nonnegative for any $s \in \mathbb{R}_+$. 

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(c) the increments $N(t_1), N(t_2) - N(t_1), \ldots, N(t_n) - N(t_{n-1})$ are independent for any $0 < t_1 < t_2 < \cdots < t_{n-1} < t_n$;

(d) the increment $N(t + s) - N(t)$ is distributed as Poisson random variable with parameter $\Lambda((t, t + s])$.

The Poisson process is homogeneous with intensity $\lambda$, iff in addition to conditions (a), (b), (c), the distribution of the increment $N(t + s) - N(t)$ depends on the value $s \in \mathbb{R}_+$ but is independent of $t \in \mathbb{R}_+$. That is, the increments are stationary.

Proof. We have already seen that definition of Poisson processes implies all four conditions. We will show in a later lecture that the above four conditions imply that $(N(t) : t \in \mathbb{R}_+)$ is a simple Poisson counting process. \qed

### A Functions with semigroup property

**Lemma A.1.** A unique non-negative right continuous function $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfying the semigroup property

$$f(t + s) = f(t)f(s), \text{ for all } t, s \in \mathbb{R}_+$$

is $f(t) = e^{\theta t}$, where $\theta = \log f(1)$.

**Proof.** Clearly, we have $f(0) = f^2(0)$. Since $f$ is non-negative, it means $f(0) = 1$. By definition of $\theta$ and induction for $m, n \in \mathbb{Z}^+$, we see that

$$f(m) = f(1)^m = e^{\theta m}, \quad e^\theta = f(1) = f(1/n)^n.$$

Let $q \in \mathbb{Q}$, then it can be written as $m/n, n \neq 0$ for some $m, n \in \mathbb{Z}^+$. Hence, it is clear that for all $q \in \mathbb{Q}^+$, we have $f(q) = e^{\theta q}$, either unity or zero. Note, that $f$ is a right continuous function and is non-negative. Now, we can show that $f$ is exponential for any real positive $t$ by taking a sequence of rational numbers $(q_n : n \in \mathbb{N})$ decreasing to $t$. From right continuity of $f$, we obtain

$$f(t) = \lim_{q_n \downarrow t} f(q_n) = \lim_{q_n \downarrow t} e^{\theta q_n} = e^{\theta t}.$$ 

\qed