## Examples of Tractable Stochastic Processes

In general, it is very difficult to characterize a stochastic process completely in terms of its finite dimensional distribution. However, we have listed few analytically tractable examples below, where we can completely characterize the stochastic process.

### 1.1 Independent and identically distributed (IID) processes

Let \((X_t : t \in T)\) be an independent and identically distributed (iid) random process, with a common distribution \(F(x)\). Then, the finite dimensional distribution for this process for any finite \(S \subseteq T\) can be written as

\[
F_S(x_S) = P\left\{ \left\{ X_s(\omega) \leq x, s \in S \right\} \right\} = \prod_{s \in S} F(x).
\]

It’s easy to verify that the first and the second moments are independent of time indices. Since \(X_t = X_0\) in distribution,

\[
m_X = \mathbb{E}X_0, \quad R_X = \mathbb{E}X_0^2, \quad C_X = \text{Var}(X_0).
\]

### 1.2 Stationary processes

A stochastic process \(X\) is stationary if all finite dimensional distributions are shift invariant. That is, for any finite \(n \in \mathbb{N}\) and \(t > 0\), the random vectors \((X_{s_1}, \ldots, X_{s_n})\) and \((X_{s_1+t}, \ldots, X_{s_n+t})\) have the identical joint distribution for all \(s_1 \leq \ldots \leq s_n\). That is, for finite \(S \subseteq T\) and \(t > 0\), we have

\[
F_S(x_S) = P\left(\left\{ X_s(\omega) \leq x, s \in S \right\}\right) = P\left(\left\{ X_{s+t}(\omega) \leq x, s \in S \right\}\right) = F_{t+S}(x_S).
\]

**Example 1.1 (IID processes).** Let \(X = X^T\) be an IID random process. Then, for any finite index subset \(S \subseteq T, t \in T\) and \(x_S \in X^S\), we can write

\[
F_S(x_S) = P\left(\left\{ X_s \leq x, s \in S \right\}\right) = \prod_{s \in S} P\left\{ X_s \leq x_s \right\} = \prod_{s \in S} P\left\{ X_{s+t} \leq x_s \right\}
\]

\[
= \prod_{u \in t+S} P\left\{ X_u \leq x_u \right\} = P\left(\left\{ X_u \leq x_u, u \in t + S \right\}\right) = F_{t+S}(x_S).
\]

First equality follows from the definition, the second from the independence of process \(X\), the third from the identical distribution for the process \(X\). In particular, we have shown that process \(X\) is also stationary.

For a stationary stochastic process, all the existing moments are shift invariant. For Gaussian random processes, first and the second moment suffice to get any finite dimensional distribution. A second order stochastic process \(X\) has finite auto-correlation \(R_X(t, t) < \infty\) for all indices \(t \in T\). This implies \(R_X(t_1, t_2) < \infty\) by Cauchy-Schwartz inequality, and hence the mean, auto-correlation, and the auto-covariance functions.
are well defined and finite. Since \( X_t = X_0 \) and \( (X_t, X_s) = (X_{t-s}, X_0) \) in distribution, we have for a second order process \( X \)

\[
m_X = \mathbb{E}X_0, \quad R_X(t, s) = R_X(t-s, 0) = \mathbb{E}X_{t-s}X_0, \quad C_X(t-s, 0) = R_X(t-s, 0) - m_X^2.
\]

A random process \( X \) is **wide sense stationary** if

1. \( m_X(t) = m_X(t + s) \) for all \( s, t \in T \), and
2. \( R_X(t, s) = R_X(t + u, s + u) \) for all \( s, t, u \in T \).

It follows that a second order stationary stochastic process \( X \), is wide sense stationary. A second order wide sense stationary process is not necessarily stationary. We can similarly define joint stationarity and joint wide sense stationarity for two stochastic processes \( X \) and \( Y \).

**Example 1.2 (Gaussian process).** Let \( X \in \mathbb{R}^R \) be a zero-mean continuous-time Gaussian process, defined by its finite dimensional distributions. In particular, for any finite \( S \subset \mathbb{R} \), we can define a column vector \( x_S \) such that \( x_S(s) = x_s \) for \( s \in S \), and the covariance matrix \( C_S \triangleq \mathbb{E}x_Sx_S^T \). Then, the finite-dimensional density is given by

\[
f_S(x_S) = \frac{1}{(2\pi)^{|S|/2}\sqrt{\det(C_S)}} \exp \left(-\frac{1}{2}x_S^TC_S^{-1}x_S\right).
\]

**Theorem 1.3.** A wide sense stationary Gaussian process is stationary.

**Proof.** Let \( X \) be a wide sense stationary Gaussian process and let \( S \subset \mathbb{R} \) be finite. From the wide sense stationarity of \( X \), we have \( \mathbb{E}X_S = 0 \) and

\[
\mathbb{E}X_{s}X_{u} = C_{s-u}, \quad \text{for all } s, u \in S.
\]

This means that \( C_S = C_{t+S} \), and the result follows.

### 1.3 Markov processes

A stochastic process \( X \) is **Markov** if conditioned on the present state, future is independent of the past. We denote the history of the process until time \( t \) as \( \mathcal{F}_t = \sigma(X_s, s \leq t) \). That is, for any ordered index set \( T \) containing any two indices \( u > t \), we have

\[
P(\{X_u \leq x_u\} \mid \mathcal{F}_t) = P(\{X_u \leq x_u\} \mid \sigma(X_t)).
\]

The range of the process is called the **state space**. We next re-write the Markov property more explicitly for the process \( X \). For all \( x, y \in X \), finite set \( S \subseteq T \) such that \( \max S < t < u \), and \( H_S = \{X_s \leq x_s : s \in S\} \in \mathcal{F}_t \), we have

\[
P(\{X_u \leq y\} \mid H_S \cap \{X_t \leq x\}) = P(\{X_u \leq y\} \mid \{X_t \leq x\}).
\]

When the state space \( X \) is countable, we can write \( H_S = \bigcap_{s \in S} \{X_s = x_s\} \) and the Markov property can be written as

\[
P(\{X_u = y\} \mid H_S \cap \{X_t = x\}) = P(\{X_u = x_u\} \mid \{X_t = x\}).
\]

We will study this process in detail in coming lectures.

**Example 1.4 (Random Walk).** Let \( X = (X_n \in \mathbb{R}^d : n \in \mathbb{N}) \) be an independent (not necessarily identical) Bernoulli sequence. Let \( S_0 = 0 \) and \( S_n \triangleq \sum_{i=1}^{n} X_i \), then the process \( S = (S_n \in \mathbb{R}^d : n \in \mathbb{N}_0) \) is called a **random walk**. We can think of \( S_n \) as the random location of a particle after \( n \) steps, where the particle starts from origin and takes steps of size \( X_i \) at the \( i \)th step.
Theorem 1.5. For a random walk \((S_n : n \in \mathbb{N})\) with independent step-size sequence \(X\), the following are true.

i. The first two moments are 
\[ \mathbb{E}S_n = \sum_{i=1}^{n} \mathbb{E}X_i \text{ and } \text{Var}[S_n] = \sum_{i=1}^{n} \text{Var}[X_i]. \]

ii. Random walk is non-stationary with independent increments. The disjoint increments are stationary if the step-size sequence \(X\) is identically distributed.

iii. Random walk is a Markov sequence.

Proof. Results follow from the independence of the step-size sequence \(X\).

i. Follows from the linearity of expectation and independence of step sizes.

ii. Since the mean is time dependent, random walk is non-stationary process. Independence of increments follows from the independence of step sizes. That is, since \(\mathcal{F}_n = \sigma(S_0, S_1, \ldots, S_n) = \sigma(S_0, X_1, \ldots, X_n)\) and the collection \((X_{n+1}, \ldots, X_m)\) is independent of \(\sigma(S_0, X_1, \ldots, X_n)\) for all \(m > n\). Since \(S_m - S_n = X_{n+1} + \cdots + X_m \in \mathcal{F}_n\), we have the independent increments. When the step-sizes are also identically distributed, the joint distributions of \((X_1, \ldots, X_{m-n})\) and \((X_{n+1}, \ldots, X_m)\) are identical. This implies the stationarity of increments for i.i.d. step-sizes.

iii. Given the historical event \(H_{n-1} \triangleq \bigcap_{k=1}^{n-1} \{S_k \leq s_k\}\) and the current state \(\{S_n \leq s_n\}\), we can write the conditional probability
\[
\mathbb{P} \left( \{S_{n+1} \leq s_{n+1}\} \mid H_{n-1} \cap \{S_n \leq s_n\} \right) = \mathbb{P} \left( \{X_{n+1} \leq s_{n+1} - S_n\} \mid H_{n-1} \cap \{S_n \leq s_n\} \right)
\]
\[
= \mathbb{P} \left( \{S_{n+1} \leq s_{n+1}\} \mid \{S_n \leq s_n\} \right).
\]
The equality in the second line follows from the independence of the step-size sequence. In particular, from the independence of \(X_{n+1}\) from the collection \(\sigma(S_0, X_1, \ldots, X_n) = \sigma(S_0, S_1, \ldots, S_n)\). For the countable state space \(X\), an given the historical event \(H_{n-1} \triangleq \bigcap_{k=1}^{n-1} \{S_k = s_k\}\) and the current state \(\{S_n = s_n\}\), we can write the conditional probability
\[
\mathbb{P} \left( \{S_{n+1} = s_{n+1}\} \mid H_{n-1} \cap \{S_n = s_n\} \right) = \mathbb{P} \left( \{X_{n+1} = s_{n+1} - S_n\} \mid H_{n-1} \cap \{S_n = s_n\} \right)
\]
\[
= \mathbb{P} \left( \{S_{n+1} = s_{n+1}\} \mid \{S_n = s_n\} \right) = \mathbb{P} \{X_{n+1} = s_{n+1} - s_n\}. \]