Lecture-18: Random Processes

1 Introduction

Let \((\Omega, \mathcal{F}, P)\) be a probability space. For an arbitrary index set \(T\) and state space \(X \subseteq \mathbb{R}\), a random process is a measurable map \(X : (\Omega, T) \to X\). Realization of random process at each \(t \in T\), is a random variable defined on the probability space \((\Omega, \mathcal{F}, P)\) as \(X_t : \Omega \to X\) such that

\[ X_t(\omega) \overset{\Delta}{=} X(\omega, t) \in X, \text{ and } X_t = (X_t(\omega) \in X : \omega \in \Omega). \]

For each outcome \(\omega \in \Omega\), we have a function \(X_\omega : T \to X\) called the sample path or the sample function of the process \(X\) defined as

\[ X_\omega(t) = X(\omega, t) \in X, t \in T. \]

The random process \(X\) can be thought of as a collection of random variables \(X = (X_t \in X : t \in T)\) or an ensemble of sample paths \(X = (X_\omega \in \mathcal{X}^T : \omega \in \Omega)\). Recall that \(\mathcal{X}^T\) is set of all functions from the index set \(T\) to state space \(X\).

Example 1.1 (Bernoulli sequence). Let index set \(T = \mathbb{N} = \{1, 2, \ldots\}\) and the sample space be the collection of infinite bi-variate sequences of successes (S) and failures (F) defined by \(\Omega = \{S, F\}^\mathbb{N}\). An outcome \(\omega \in \Omega\) is an infinite sequence \(\omega = (\omega_1, \omega_2, \ldots)\) such that \(\omega_n \in \{S, F\}\) for each \(n \in \mathbb{N}\). We define the random process \(X : (\Omega, \mathbb{N}) \to \{0, 1\}\) such that \(X_\omega = (1_{\{\omega_1 = S\}}, 1_{\{\omega_2 = S\}}, \ldots)\). That is, we have

\[ X(\omega, n) = 1_{\{\omega_n = S\}}, \quad X_n = (1_{\{\omega_n = S\}} : \omega_n \in \{S, F\}), \quad X_\omega = (1_{\{\omega_n = S\}} : n \in \mathbb{N}). \]

Hence, we can write the process as collection of random variables \(X = (X_n : n \in \mathbb{N})\) or the collection of sample paths \(X = (X_\omega : \omega \in \Omega)\).

1.1 Classification

State space \(X\) can be countable or uncountable, corresponding to discrete or continuous valued process. If the index set \(T\) is countable, the stochastic process is called discrete-time stochastic process or random sequence. When the index set \(T\) is uncountable, it is called continuous-time stochastic process. The index set \(T\) doesn’t have to be time, if the index set is space, and then the stochastic process is spatial process. When \(T = \mathbb{R}^n \times [0, \infty)\), stochastic process \(X\) is a spatio-temporal process.

Example 1.2. We list some examples of each such stochastic process.

i_ Discrete random sequence: brand switching, discrete time queues, number of people at bank each day.

ii_ Continuous random sequence: stock prices, currency exchange rates, waiting time in queue of \(n\)th arrival, workload at arrivals in time sharing computer systems.

iii_ Discrete random process: counting processes, population sampled at birth-death instants, number of people in queues.
Continuous random process: water level in a dam, waiting time till service in a queue, location of a mobile node in a network.

1.2 Independence

Recall, given the probability space \((\Omega, \mathcal{F}, P)\), two events \(A, B \in \mathcal{F}\) are independent events if
\[
P(A \cap B) = P(A)P(B).
\]
Random variables \(X, Y\) defined on the above probability space, are independent random variables if for all \(x, y \in \mathbb{R}\)
\[
P\{X(\omega) \leq x, Y(\omega) \leq y\} = P\{X(\omega) \leq x\}P\{Y(\omega) \leq y\}.
\]
A stochastic process \(X\) is said to be independent if for all finite subsets \(S \subseteq T\), the finite collection of events \(\{X_s \leq x_s, s \in S\}\) are independent. That is, we have
\[
P(\{X_s \leq x_s, s \in S\}) = \prod_{s \in S} P\{X_s \leq x_s\}.
\]
Two stochastic processes \(X, Y\) for the common index set \(T\) are independent random processes if for all finite subsets \(I, J \subseteq T\), the following events \(\{X_i \leq x_i, i \in I\}\) and \(\{Y_j \leq y_j, j \in J\}\) are independent. That is,
\[
P(\{X_i \leq x_i, i \in I\} \cap \{Y_j \leq y_j, j \in J\}) = P(\{X_i \leq x_i, i \in I\})P(\{Y_j \leq y_j, j \in J\}).
\]

Example 1.3 (Bernoulli sequence). Let the Bernoulli sequence \(X\) defined in Example 1.1 be independent and identically distributed with \(P\{X_i = 1\} = p \in (0, 1)\). For any sequence \(x \in \{0, 1\}^\mathbb{N}\), we have
\[P\{X = x\} = 0\] Let \(q \triangleq (1 - p)\), then the probability of observing \(m\) heads and \(r\) tails is given by \(p^m q^r\).

1.3 Specification

To define a measure on a random process, we can either put a measure on sample paths, or equip the collection of random variables with a joint measure. We are interested in identifying the joint distribution \(F : \mathbb{R}^T \to [0, 1]\). To this end, for any \(x \in \mathbb{R}^T\) we need to know
\[
F(x) = P(\bigcap_{t \in T} \{\omega \in \Omega : X_t(\omega) \leq x_t\}) = P(\bigcap_{t \in T} X_t^{-1}(-\infty, x_t]) = P \circ X^{-1} \big(X^T \cap (-\infty, x_T]\big).
\]
However, even for a simple independent process with countably infinite \(T\), any function of the above form would be zero if \(x_t\) is finite for all \(t \in T\). Therefore, we only look at the values of \(F(x)\) when \(x_t \in \mathbb{R}\) for indices \(t\) in a finite set \(S\) and \(x_t = \infty\) for all \(t \notin S\). That is, for any finite set \(S \subseteq T\) we focus on the product sets of the form
\[
\bigtimes_{s \in S} (-\infty, x_s] \times \mathbb{R}.
\]
We can define a finite dimensional distribution for any finite set \(S \subseteq T\) and \(x_s = \{x_s \in \mathbb{R} : s \in S\}\),
\[
F_S(x_S) = P\left(\bigcap_{s \in S} \{\omega \in \Omega : X_s(\omega) \leq x_s\}\right) = P\left(\bigcap_{s \in S} X_s^{-1}(-\infty, x_s]\right).
\]
Set of all finite dimensional distributions of the stochastic process \(\{X_t : t \in T\}\) characterizes its distribution completely. Simpler characterizations of a stochastic process \(X(t)\) are in terms of its moments. That is, the first moment such as mean, and the second moment such as correlations and covariance functions.
\[
m_X(t) \triangleq \mathbb{E}X_t, \quad R_X(t, s) \triangleq \mathbb{E}X_tX_s, \quad C_X(t, s) \triangleq \mathbb{E}(X_t - m_X(t))(X_s - m_X(s)).
\]
Example 1.4 (Bernoulli sequence). We can easily compute the mean, the auto-correlation, and the auto-covariance functions for the independent Bernoulli process defined in Example 1.3 as

\[ m_X(n) = \mathbb{E}X_n = p, \quad R_X(m,n) = \mathbb{E}X_mX_n = \mathbb{E}X_m\mathbb{E}X_n = p^2, \quad C_X(m,n) = 0. \]

Example 1.5. Some examples of simple stochastic processes.

i. \( X_t = A \cos 2\pi t \), where \( A \) is random.

ii. \( X_t = \cos(2\pi t + \Theta) \), where \( \Theta \) is random and uniformly distributed between \((-\pi, \pi]\).

iii. \( X_n = U^n \) for \( n \in \mathbb{N} \), where \( U \) is uniformly distributed in the open interval \((0,1)\).

iv. \( Z_t = At + B \) where \( A \) and \( B \) are independent random variables.