Lecture 13: Foster-Lyapunov Theorem

1 Foster’s Theorem

**Theorem 1.1 (Foster, 1950).** Let \( \{X_n\}_{n \geq 0} \) be a irreducible DTMC on \( \mathbb{N}_0 \) if there exist a function \( L : \mathbb{N}_0 \rightarrow \mathbb{R}_+ \) with \( \mathbb{E}[L(X_0)] < \infty \), such that for some \( K > k \geq 0 \), and \( \epsilon > 0 \):

1. \( |\{x \in \mathbb{N}_0 : L(x) \leq k\}| < \infty \)
2. \( \mathbb{E}[L(X_n)|X_{n-1}] < K \), when \( L(X_{n-1}) \leq k \).
3. \( \mathbb{E}[L(X_n) - L(X_{n-1})|X_{n-1}] < -\epsilon \) if \( L(X_{n-1}) \geq k \).

Then \( \{X_n\}_{n \geq 0} \) is positive recurrent. (\( L \equiv \) "potential function" or energy or lyapunov function).

**Proof.** As DTMC is irreducible than enough to show that some state is positive recurrent. By renewal theory, for ant DTMC, for all \( x \in \mathbb{N}_0 \),

\[
\lim_{N \to \infty} \mathbb{E}\left[ \sum_{n=1}^{N} 1_{\{X_n = x\}} \right] = \frac{1}{\mu_{xx}} \tag{1}
\]

where

\[
\mu_{xx} = \begin{cases} 
\infty & \text{if } x \text{ is transient} \\
\sum_{m \geq 0} m f_{xx} & \text{if } x \text{ is recurrent}
\end{cases}
\]

consider the RHS of equation (1)

\[
\lim_{N \to \infty} \mathbb{E}\left[ \sum_{n=1}^{N} 1_{\{X_n = x\}} \right] > 0 \iff x \text{ is positive recurrent}
\]
consider
\[
0 \leq \mathbb{E}[L(X_n)] = \mathbb{E}[L(X_0)] + \sum_{n=1}^{N} \mathbb{E}[L(X_n) - L(X_{n-1})]
\]
\[
= \mathbb{E}[L(X_0)] + \sum_{n=1}^{N} \mathbb{E}[L(X_n) - L(X_{n-1})]1\{L(X_{n-1}) > k\} + \sum_{n=1}^{N} \mathbb{E}[L(X_n) - L(X_{n-1})]1\{L(X_{n-1}) \leq k\}
\]
\[
\leq \mathbb{E}[L(X_0)] + \sum_{n=1}^{N} \mathbb{E}[-\epsilon 1\{L(X_{n-1}) > k\}] + \sum_{n=1}^{N} \mathbb{E}[K 1\{L(X_{n-1}) \leq k\}]
\]
\[
\Rightarrow \left( \mathbb{E} \sum_{n=1}^{N} 1\{L(X_{n-1}) \leq k\} \right) (K + \epsilon) \geq -\mathbb{E}[L(X_0)] + \epsilon N
\]
\[
\Rightarrow \frac{1}{N} \left( \mathbb{E} \sum_{n=1}^{N} 1\{L(X_{n-1}) \leq k\} \right) \geq -\frac{\mathbb{E}[L(X_0)]}{K + \epsilon} + \frac{\epsilon}{K + \epsilon}
\]
\[
\Rightarrow \limsup_{N \to \infty} \frac{1}{N} \left( \mathbb{E} \sum_{n=1}^{N} 1\{L(X_{n-1}) \leq k\} \right) \geq \frac{\epsilon}{K + \epsilon}
\]

Let \( F = \{ x \in \mathbb{N}_0 : L(x) \leq k \} \), \(|F| < \infty\). Now,

\[
\limsup_{N \to \infty} \frac{1}{N} \left( \mathbb{E} \sum_{n=1}^{N} 1\{L(X_{n-1}) \leq k\} \right) = \limsup_{N \to \infty} \frac{1}{N} \left( \mathbb{E} \sum_{n=1}^{N} \sum_{x \in F} 1\{X_{n-1} \leq k\} \right) \leq \sum_{x \in F} \left( \limsup_{N \to \infty} \frac{1}{N} \mathbb{E} \sum_{n=1}^{N} 1\{X_{n-1} \leq k\} \right)
\]
\[
= \sum_{x \in F} \left( \limsup_{N \to \infty} \frac{1}{N} \mathbb{E} \sum_{n=1}^{N} 1\{X_{n-1} \leq k\} \right) \geq \frac{\epsilon}{K + \epsilon} > 0
\]

Therefore there exist some \( x \in F \) such that ,

\[
\limsup_{N \to \infty} \frac{1}{N} \mathbb{E} \left( \sum_{n=1}^{N} 1\{X_{n-1} \leq k\} \right) \geq \frac{\epsilon}{(K + \epsilon)|F|} > 0
\]

Therefore there exist some \( x \in F \) such that ,

\[
\lim_{N \to \infty} \frac{1}{N} \mathbb{E} \left( \sum_{n=1}^{N} 1\{X_{n-1} \leq k\} \right) > 0
\]
2 Applications of Foster’s theorem: Queue scheduling/Max-weight scheduling

Consider $N$ queue served by a single server in discrete time (Figure 1). At time slot $t = 1,2,3,...$, $A_i(t) \in \mathbb{N}_0$ packets arrive to each queue $i \in [N]$ independently.

1. $E[A_i(t)] = \lambda_i$
2. $P[A_i(t) = 0] > 0$
3. $E[A_i(t)^2] \leq C$

Server picks one queue $Q(t) \in [N]$ for service. Let $R_i(t) = 1\{Q(t) = i\}$. One packet is served from $Q(t)$ if it is not empty. Let $X_i(t)$ = number of packets in queue $i$ just before time slot $t$.

$$X_i(t+1) = (X_i(t) + A_i(t) - R_i(t))_+$$

Where

$$a_+ = \max(0,a)$$
\[ X_i(t+1) = X_i(t) + A_i(t) - R_i(t) + L_i(t) \]

Where

\[ L_i(t) = \begin{cases} 1 & \text{service attempted when } i \text{ is empty} \\ 0 & \text{otherwise} \end{cases} \]

Mean rate of arrivals to system: \( \sum_{i=1}^{N} \lambda_i \)

Maximum rate of departure \( = 1 \), we will assume \( \sum_{i=1}^{N} \lambda_i < 1 \).

**Theorem 2.1 (Max-weight scheduling algorithm, 1992).**

\[ Q(t) = \arg \max_{i \in N} X_i(t) \]

that is serve the longest queue. Under MAX-WT, \( X(t) = (X_i(t))_{i=1}^{N} \) is a DTMC which is irreducible and aperiodic on state space \( \mathbb{N}_0^N \). As long as \( \sum_{i=1}^{N} \lambda_i < 1 \), \( \{X_n\} \) is positive recurrent.

**Proof.** By Foster’s theorem, define the Lyapunov function:

\[ L(x) = \frac{1}{2} \sum_{i=1}^{N} x_i^2 \]

consider

\[
L(X(t)) - L(X(t-1)) \\
= \frac{1}{2} \sum_{i=1}^{N} \left[ (X_i(t))^2 - (X_i(t-1))^2 \right] \\
= \frac{1}{2} \sum_{i=1}^{N} \left[ (X_i(t) - X_i(t-1))^2 \right] \\
\leq \frac{1}{2} \sum_{i=1}^{N} \left[ (A_i(t) - R_i(t))^2 - (X_i(t-1))^2 \right]
\]

Therefore

\[
\mathbb{E}\left[ L(X(t)) - L(X(t-1))|X(t-1) = x \right] \leq \frac{1}{2} \sum_{i=1}^{N} \mathbb{E}\left[ (x_i + A_i(t-1) - R_i(t))^2 - x_i^2 |X(t-1) = x \right] \\
= \frac{1}{2} \sum_{i=1}^{N} \left[ 2x_i \mathbb{E}\left[ (A_i(t-1) - R_i(t))|X(t-1) = x \right] + \mathbb{E}\left[ (A_i(t-1) - R_i(t-1))^2 |X(t-1) = x \right] \right] \\
= \sum_{i=1}^{N} x_i \lambda_i - \sum_{i=1}^{N} x_i R_i(t-1) + \frac{N}{2} (1 + C) \\
= \sum_{i=1}^{N} x_i \lambda_i + \frac{N}{2} (1 + C) - \max_{i} x_i \\
\leq \frac{N}{2} (1 + C) + (\max_{i} x_i) \left( \sum_{i=1}^{N} \lambda_i - 1 \right) \\
= C_1 - \epsilon (\max_{i} x_i)
\]
Foster’s theorem applies with $k = \max\left\{ \frac{\|x\|^2}{2} \right\}$.