1 Renewal theory Contd. – Delayed Renewal processes

1.1 Example: (Optional – not covered in class)

Consider two coins and suppose that each time is coin flipped, it lands tail with some unknown probability \( p_i, \ i = 1, 2 \). We are interested in coming up with a strategy that ensures that long term proportion of tails is \( \min\{p_1, p_2\} \). One strategy is as follows: Set \( n = 1 \). In the \( n \)th round of coin flipping, flip the first coin till \( n \) consecutive tails are obtained. Then flip the second coin till \( n \) consecutive tails are obtained. Increment \( n \) and repeat.

**Claim.** \( \lim_{m \to \infty} \frac{\text{#tails in the first } m \text{ tosses}}{m} = \min\{p_1, p_2\} \) with probability 1.

The proof is as follows. Let \( p = \max\{p_1, p_2\} \) and \( \alpha p = \min\{p_1, p_2\} \). There is nothing to prove if \( \alpha = 1 \), so let \( \alpha < 1 \). Call the coin with \( P(T) = p \), the bad coin and the other, the good coin. Let \( B_n \) denote the number of flips in the \( n \)th round of tossing the bad coin, and \( G_n \) the number of flips in the \( n \)th round of tossing the good coin. We first prove the following lemma.

**Lemma 1.1.** For any \( \epsilon > 0 \) with \( \epsilon^{-1} \in \mathbb{N} \), \( P(B_n \geq \epsilon G_n \text{ for infinitely many rounds } n) = 0 \).

**Proof.** For any \( n \in \mathbb{N} \),
\[
P \left( G_n \leq \frac{B_n}{\epsilon} \right) = \mathbb{E}[P(G_n \leq \frac{B_n}{\epsilon}|B_n)]
\]
\[
= \mathbb{E}\left[ \sum_{i=1}^{B_n} P(G_n = i|B_n) \right]
\]
\[
\leq \mathbb{E}\left[ \sum_{i=1}^{B_n} (\alpha p)^n \right]
\]
\[
= \mathbb{E}\left[ \frac{B_n}{\epsilon} \right] (\alpha p)^n
\]
\[
= \epsilon^{-1} \left( \sum_{i=1}^{\infty} \frac{1}{p^i} \right) (\alpha p)^n = \epsilon^{-1} \frac{p^{-n} - 1}{1-p} (\alpha p)^n,
\]

where the inequality follows from the fact that \( \{G_m = i\} \) implies that \( i \geq m \) and that in cycle \( m \), the coin flips numbered \( i - m + 1 \) to \( i \) are all tails. Hence,
\[
\sum_{n=1}^{\infty} P \left( G_n \leq \frac{B_n}{\epsilon} \right) \leq \epsilon^{-1} \sum_{n=1}^{\infty} \frac{\alpha^n}{1-p} < \infty.
\]
By the Borel-Cantelli lemma, it follows that $P(B_n \geq \epsilon G_n \text{for infinitely many } n) = 0$. \hfill \square

With probability 1, all but a finite number of rounds have at most an $\epsilon$ fraction of bad coin tosses, implying that $\lim_{m \to \infty} \# \text{bad coin tosses in the first } m \text{ tosses} = \epsilon$. Now taking a decreasing sequence $\epsilon_k = 1/k$, $k = 1, 2, 3, \ldots$, and using the continuity of probability, we get that with probability 1, $\lim_{m \to \infty} \# \text{bad coin tosses in the first } m \text{ tosses} = 0$. This proves the claim using the strong law of large numbers for tosses of the good coin.

1.2 Distribution of the Last Renewal Time for a Delayed Renewal Process

In the same manner as we derived the key lemma, refer Theorem 1.9 in lecture 6, for the last renewal time distribution of a standard renewal process, we can show for a delayed renewal process:

$$P(S_N(t) \leq s) = G^c(t)P(S_N(t) \leq s|S_N(t) = 0) + \int_0^t P(S_N(t) \leq s|S_N(t) = u)F^c(t-u)dm(u)$$

= $G^c(t) + \int_0^s F^c(t-u)dm(u)$.

Let $F_e(x) = \frac{\int_0^x F(y)dy}{\mu}$, $x \geq 0$, known as the equilibrium distribution of $F$. Observe that the moment generating function of $F_e(x)$ is $\tilde{F}_e(s) = \frac{1-\tilde{F}(s)}{s\mu}$.

**Proof.** By definition, $\tilde{F}_e(s) = E[e^{-sX}]$, where $X$ is a random variable with probability distribution function $F_e(x)$. So,

$$\tilde{F}_e(s) = \int_0^\infty e^{-sx}dF_e(x)$$

= $\frac{1}{\mu} \int_0^\infty e^{-sx}F^c(x)dx$

= $\frac{1}{s\mu} - \frac{1}{\mu} \int_0^\infty e^{-sx}F(x)dx$

= $\frac{1}{s\mu} - \frac{1}{s\mu} \int_0^{\infty} e^{-sx}dF(x)$

= $\frac{1}{s\mu} - \frac{1}{s\mu} \tilde{F}(s)$,

where the third and fourth equalities follows from the basic integration techniques. \hfill \square

And also observe that $F_e$ is the limiting distribution of the age and the excess time for the renewal process governed by $F$. If $G = F_e$, then the delayed renewal process is called the equilibrium renewal process. Suppose we start observing a renewal process at some arbitrary time $t$. Then, the observed renewal process is the equilibrium renewal process. Let $Y_e(t)$ denote the excess time for the (delayed) equilibrium renewal process.

**Theorem 1.2.** For the equilibrium renewal process,

1. $m_e(t) = \frac{t}{\mu}$. 

2.
2. \( P(Y_e(t) \leq x) = F_e(x) \).

3. \( \{N_e(t), t \geq 0\} \) has stationary increments.

Proof. To prove (1), observe that 
\[
\tilde{m}_e(s) = \frac{\hat{G}(s)}{1-F(s)} = \frac{\hat{F}_e(s)}{1-F(s)} = \frac{1}{s\mu}. 
\]
Hence, if \( m_e(t) = \frac{t}{\mu} \) then,
\[
\tilde{m}_e(s) = \int_0^\infty e^{-st}dm_e(t)
= \frac{1}{\mu} \int_0^\infty e^{-st}dt
= \frac{1}{s\mu}.
\]
Since moment generating function is a one-to-one map, \( m_e(t) = \frac{t}{\mu} \) is unique.

(2)
\[
P(Y_e(t) > x) = P(Y_e(t) > x|S_{N_e(t)} = 0)P(S_{N_e(t)} = 0) + P(Y_e(t) > x|S_{N_e(t)} = s) F_e(t - s) \frac{ds}{\mu}
= P(X_1 > t + x, X_1 > t) + P(X_2 > t + x - s|X_2 > t - s) F_e(t - s) \frac{ds}{\mu}
= F_e(x + t) + \int_0^t F_e(t + x - s) \frac{ds}{\mu}
= 1 - \frac{1}{\mu} \int_0^{t+x} F_e(y)dy - \frac{1}{\mu} \int_{t+x}^x F_e(y)dy
= 1 - \frac{1}{\mu} \int_0^x F_e(y)dy
= F_e(x).
\]

(3) \( N_e(t+s) - N_e(s) \) = Number of renewals in time interval of length \( t \). When we start observing at \( s \), the observed renewal process is delayed renewal process with initial distribution being the original distribution. 

Question: What can you say about the equilibrium renewal process when \( F \) is distributed exponentially with the parameter \( \lambda \)?

Answer: Let’s look at the distribution of the first inter-arrival distribution, \( F_e \). So,
\[
F_e(x) = \frac{1}{\mu} \int_0^x F_e(y)dy
= \lambda \int_0^x e^{-y\lambda}dy
= 1 - e^{-x\lambda},
\]
where the first equality follows from the definition of \( F_e \) for equilibrium renewal process, the second equality follows from the fact that the mean of exponential distribution is inverse of the parameter \( \lambda \).

Thus even \( F_e \) is distributed exponentially with the parameter \( \lambda \). So with all the properties of equilibrium renewal process, \( F_e \) and \( F \) being distributed exponentially with the same parameter \( \lambda \), says that this is a poisson process (not a delayed renewal process).
1.3 Renewal Reward Process

**Definition:** Consider a renewal process \( \{N(t), t \geq 0\} \) with inter arrival times \( \{X_n : n \in \mathbb{N}\} \) having distribution \( F \) and rewards \( \{R_n : n \in \mathbb{N}\} \) where \( R_n \) is the reward at the end of \( X_n \). Let \( (X_n, R_n) \) be iid. Then \( R(t) = \sum_{i=1}^{N(t)} R_i \) is reward process (total reward earned by time \( t \)).

**Theorem 1.3.** Let \( \mathbb{E}[|R|] \) and \( \mathbb{E}[|X|] \) be finite.

1. \( \lim_{t \to \infty} \frac{R(t)}{t} = \frac{\mathbb{E}[R]}{\mathbb{E}[X]} \) a.s.
2. \( \lim_{t \to \infty} \frac{\mathbb{E}[R(t)]}{t} = \frac{\mathbb{E}[R]}{\mathbb{E}[X]} \).

**Proof.** (1) Write
\[
R(t) = \frac{\sum_{i=1}^{N(t)} R_i}{t} = \left( \frac{\sum_{i=1}^{N(t)} R_i}{N(t)} \right) \left( \frac{N(t)}{t} \right).
\]

By the strong law of large numbers (almost sure convergence law) we obtain that,
\[
\lim_{t \to \infty} \frac{\sum_{i=1}^{N(t)} R_i}{N(t)} = \mathbb{E}[R],
\]
and by the basic renewal theorem (almost sure convergence law) we obtain that,
\[
\lim_{t \to \infty} \frac{N(t)}{t} = \frac{1}{\mathbb{E}[X]}.
\]

Thus (1) is proven.

(2) Notice that \( N(t) + 1 \) is a stopping time for the sequence \( \{R_1, R_2, \ldots\} \). This is true since
\[
\{N(t) + 1 = n\} = \{X_1 + X_2 + \cdots + X_{n-1} \leq t, X_n > t\} = \{R_1 + R_2 + \cdots + R_{n-1} = R(t), R_n \neq 0\}.
\]

Moreover \( N(t) + 1 \) is a stopping time for the sequence \( \{X_1, X_2, \ldots\} \). So by algebra and Wald’s lemma,
\[
\mathbb{E}[R(t)] = \mathbb{E} \left[ \sum_{i=1}^{N(t)} R_i \right] = \mathbb{E} \left[ \sum_{i=1}^{N(t)+1} R_i \right] - \mathbb{E}[R_{N(t)+1}] = (m(t) + 1)\mathbb{E}[R_1] - \mathbb{E}[R_{N(t)+1}].
\]

Let \( g(t) = \mathbb{E}[R_{N(t)+1}]. \) So
\[
\frac{\mathbb{E}[R(t)]}{t} = \frac{(m(t) + 1)\mathbb{E}[R_1] - g(t)}{t}.
\]
and the result will follow from the elementary renewal theorem if we can show that \( \frac{g(t)}{t} \to 0 \) as \( t \to \infty \). So,

\[
g(t) = \mathbb{E}[R_{N(t)+1}|S_{N(t)} = 0] + \mathbb{E}[R_{N(t)+1}|S_{N(t)} > 0]
\]

\[
= \mathbb{E}[R_{N(t)+1}|S_{N(t)} = 0]P(X_1 > t) + \int_0^t \mathbb{E}[R_{N(t)+1}|S_{N(t)} = u]F^c(t-u)dm(u),
\]

where the second equality follows from the fact that the interarrival times \( X_n, n \in \mathbb{N}, \) are iid with distribution \( F \).

However,

\[
\mathbb{E}[R_{N(t)+1}|S_{N(t)} = 0] = \mathbb{E}[R_1|X_1 > t],
\]

\[
\mathbb{E}[R_{N(t)+1}|S_{N(t)} = u] = \mathbb{E}[R_n|X_1 > t-u],
\]

and so

\[
g(t) = \mathbb{E}[R_1|X_1 > t]F^c(t) + \int_0^t \mathbb{E}[R_n|X_1 > t-u]F^c(t-u)dm(u)
\]

\[
= \mathbb{E}[R_1|X_1 > t]F^c(t) + \int_0^t \mathbb{E}[R_1|X_1 > t-u]F^c(t-u)dm(u),
\]

where the second equality follows from the fact that \( R_n, n \in \mathbb{N}, \) are iid.

Now, let

\[
h(t) = \mathbb{E}[R_1|X_1 > t]F^c(t) = \int_{x=t}^{\infty} \mathbb{E}[R_1|X_1 = x]dF(x),
\]

and note that since

\[
\mathbb{E}[|R_1|] = \int_{x=0}^{\infty} \mathbb{E}[|R_1||X_1 = x]dF(x) < \infty,
\]

it follows that \( h(t) \to 0 \) as \( t \to \infty \). Hence, choosing \( T \) such that \( |h(u)| \leq \epsilon \) whenever \( u \geq T \), we have for all \( t \geq T \) that

\[
\frac{|g(t)|}{t} \leq \frac{|h(t)|}{t} + \int_0^{t-T} \frac{|h(t-s)|}{t}dm(s) + \int_{t-T}^t \frac{|h(t-s)|}{t}dm(s)
\]

\[
\leq \frac{\epsilon}{t} + \frac{cm(t-T)}{t} + \mathbb{E}[|R_1|]\left(\frac{m(t) - m(t-T)}{t}\right).
\]

Hence \( \lim_{t \to \infty} \frac{g(t)}{t} = \frac{\epsilon}{E[X]} \) by the elementary renewal theorem, and the result follows since \( \epsilon > 0 \) is arbitrary.

**Remark 1.4.** (1) \( R_{N(t)+1} \) has different distribution than \( R_1 \).

Analysis: Notice that \( R_{N(t)+1} \) is related to \( X_{N(t)+1} \) which is the length of the renewal interval containing the point \( t \). Since larger renewal intervals have a greater chance of containing \( t \), it follows that \( X_{N(t)+1} \) tends to be larger than a ordinary renewal interval. Formally,

\[
\Pr\{X_{N(t)+1} > x\} = \sum_{n \in \mathbb{N}_0} \left( \int_0^t \Pr\{X_{N(t)+1} > x|S_{N(t)} = y, N(t) = n\}F^c(t-y)dm(y) \right) \Pr\{N(t) = n\}.
\]
Now we have,

\[
\Pr\{X_{N(t)+1} > x | S_{N(t)} = y, N(t) = n\} = \Pr\{X_{N(t)+1} > x | X_1 + \cdots + X_n = y, X_{n+1} > t - y\}
\]
\[
= \Pr\{X_{n+1} > x | X_{n+1} > t - y\}
\]
\[
= \frac{\Pr\{X_{n+1} > \max(x, t - y)\}}{\Pr\{X_{n+1} > t - y\}}
\]
\[
\geq F^c(x).
\]

So we get that,

\[
\Pr\{X_{N(t)+1} > x\} \geq \Pr\{X_1 > x\}.
\]

Thus the remark follows.

(2) \( R(t) \) is the gradual reward during a cycle,

\[
\frac{\sum_{n=1}^{N(t)} R_n}{t} \leq \frac{R(t)}{t} \leq \frac{\sum_{n=1}^{N(t)+1} R_n}{t}.
\]

**Analysis:** The part 1 of the theorem \[1.3\] under this regime follows since

\[
\lim_{t \to \infty} \frac{\sum_{n=1}^{N(t)} R_n}{t} = \frac{\mathbb{E}[R]}{\mathbb{E}[X]},
\]
\[
\lim_{t \to \infty} \frac{\sum_{n=1}^{N(t)+1} R_n}{t} = \frac{\mathbb{E}[R]}{\mathbb{E}[X]}.
\]

by the similar arguments given in the proof of the theorem \[1.3\].

The part 2 of the theorem \[1.3\] under this regime follows since

\[
\lim_{t \to \infty} \frac{\mathbb{E}[R_{N(t)+1}]}{t} = 0,
\]

by the similar arguments given in the proof of the theorem \[1.3\]. Thus the remark follows. For more insights refer Chapter 3 in *Stochastic Processes* by Sheldon M. Ross.

**1.3.1 Example:**

Suppose for an alternating renewal process, we earn at a rate of one per unit time when the system is on and the reward for a cycle is the time system is ON during that cycle.

\[
\lim_{t \to \infty} \frac{\text{Amount of ON time in } [0, t]}{t} = \lim_{t \to \infty} \frac{R(t)}{t} = \frac{\mathbb{E}[X]}{\mathbb{E}[X] + \mathbb{E}[Y]} = \lim_{t \to \infty} P(\text{ON at time } t).
\]