1 Age-dependent Branching Process

Suppose an organism lives up to a time period of $X \sim F$ and produces $N \sim P$ number of offspring. Let $X(t)$ denote the number of organisms alive at time $t$. The stochastic process \{X(t), t \geq 0\} is called an age-dependent branching process. We are interested in computing $M(t) = \mathbb{E}[X(t)]$ when $m = \mathbb{E}[N] = \sum_{j \in \mathbb{N}} j P_j$.

**Theorem 1.1.** If $X(0) = 1$, $m > 1$ and $F$ is non-lattice, then

$$
\lim_{t \to \infty} e^{-\alpha t} M(t) = \frac{m - 1}{m^2 \alpha \int_0^\infty x e^{-\alpha x} dF(x)},
$$

where $\alpha > 0$ is unique such that $\int_0^\infty x e^{-\alpha x} dF(x) = \frac{1}{m}$.

**Proof.** Condition on $T_1$, the life time of first organism,

$$
M(t) = \int_0^\infty \mathbb{E}[X(t)|T_1 = y] dF(y) \tag{2}
$$

\begin{align*}
\overset{(a)}{=} & \int_0^t 1 dF(y) + \int_y^\infty m M(t - y) dF(y).
\end{align*}

Thus we get

$$
M(t) = F^c(t) + m \int_0^t M(t - y) dF(y) \tag{1}
$$

Let $\alpha$ denote the unique positive number such that $\int_0^\infty x e^{-\alpha x} dF(x) = \frac{1}{m}$ and $G(y) = m \int_0^y e^{-\alpha y} dF(y)$. Upon multiplying both sides of equation (2) by $e^{-\alpha t}$ and defining $f(t) = e^{-\alpha t} M(t)$, $h(t) = e^{-\alpha t} F^c(t)$,

$$
f = h + f \ast G = h + G \ast (h + f \ast G) \tag{3}
$$

$$
\overset{\ddots}{=} h + h \ast \sum_{i=1}^{\infty} G_i = h + h \ast m G.
$$
Or, \( f(t) = h(t) + \int_0^t h(t-s)dm_G(s) \). It can be shown that \( h(t) \) is dRi and hence by Key Renewal thmrem,

\[
f(t) \to \frac{\int_0^\infty e^{-\alpha t}F^c(t)dt}{\int_0^\infty xG(x)}.
\]

\[
\int_0^\infty e^{-\alpha t}F^c(t)dt = \int_0^\infty e^{-\alpha t} \int_t^\infty dF(x)dt
\]

\[
= \int_0^\infty \int_0^x e^{-\alpha t}dtdF(x)
\]

\[
= \int_0^\infty (1 - e^{-\alpha x})dF(x)
\]

\[
= \frac{1}{\alpha} (1 - \frac{1}{\mu}) \quad \text{by the definition of } \alpha.
\]

Also \( \int_0^\infty xG(x) = m \int_0^\infty xe^{-\alpha x}dF(x) \). Hence the result follows.

2 Delayed Renewal Process

Let \( \{X_n : n \in \mathbb{N}\} \) be independent but \( X_1 \sim G \) and \( X_i \sim F, \quad i \geq 2 \) then the counting process \( \{N_D(t) : t \geq 0\} \) is called general renewal process or delayed renewal process. Let \( S_0 = 0 \) and \( S_n = \sum_{i=1}^n X_i \). We have

\[
N_D(t) = \sup \{n \in \mathbb{N} : S_n \leq t\},
\]

\[
P(N_D(t) = n) = P(S_n \leq t) - P(S_{n+1} \leq t)
\]

\[
= G * F^{n-1}(t) - G * F^n(t),
\]

\[
m_D(t) = \mathbb{E}[N_D(t)] = \sum_{n \in \mathbb{N}} G * F^{n-1}(t).
\]

Taking the Laplace transform of \( m_D(t) \), denoted as \( \tilde{m}_D(s) = \frac{\tilde{G}(s)}{1 - \tilde{F}(s)} \).

Proposition 2.1. The following holds:

1. \( \lim_{t \to \infty} \frac{N_D(t)}{t} = \frac{1}{\mu} \).
2. \( \lim_{t \to \infty} \frac{m_D(t)}{t} = \frac{1}{\mu} \).
3. If \( F \) is non-lattice, \( \lim_{t \to \infty} n_D(t + a) - m_D(t) = \frac{a}{\mu_F} \).
4. If \( F \) and \( G \) are lattice with period \( d \), \( \mathbb{E}[\# \text{of renewals at nd}] = \frac{d}{\mu_F} \).
5. If \( F \) is nonlattice, \( \mu < \infty \) and \( h \) dRi, then

\[
\lim_{t \to \infty} \int_0^t h(t-x)dm_D(x) = \frac{\int_0^\infty h(t)dt}{\mu}.
\]
2.0.1 Example:

Let \( \{X_n : n \in \mathbb{N}\} \) be iid discrete observed. A pattern \( x_1, x_2 \ldots x_k \) is said to occur at time \( n \) if \( X_n = x_k, X_{n-1} = x_{k-1}, \ldots X_{n-k+1} = x_1 \). If we have iid tosses and consider \( N(n) \) as the number of times pattern 0, 1, 0, 1 appear in \( n \) tosses, with \( P(H) = p = 1 - q \), the process is a delayed renewal processes. To find the mean number of tosses for the first time the pattern 0, 1, 0, 1 appear,

\[
E[\text{first time pattern } 0, 1, 0, 1 \text{ appears}] = E[\text{first time pattern } 0, 1 \text{ appears}] \\
+ E[\text{time between patterns } 0, 1, 0, 1] \\
= p^{-1}q^{-1} + p^{-2}q^{-2}.
\]

Similarly we can show that \( E[\text{first time } k \text{ heads}] = \sum_{i=1}^{n} p^{-i} \).