Lecture 03: Properties of Poisson Process

1 Conditional Distribution of Arrivals

**Proposition 1.1.** Let \( \{ N(t) \in [0, \infty) : t \in [n] \} \) be a Poisson process with \( \{ A_i \subseteq \mathbb{R}^+ : i \in [n] \} \) a set of finite disjoint intervals with \( B = \bigcup_{i \in [n]} A_i \), and \( \{ k_i \in \mathbb{N} : i \in [n] \} \) and \( k = \sum_{i \in [n]} k_i \). Then, we have

\[
\Pr \bigcap_{i \in [n]} \{ N_{A_i} = k_i | N(B) = k \} = k! \prod_{i \in [n]} \frac{1}{k_i!} \left( \frac{|A_i|}{|B|} \right)^{k_i}.
\]

**Proof.** It follows from the stationary independent increment property of Poisson processes that

\[
\Pr \bigcap_{i \in [n]} \{ N_{A_i} = k_i | N(B) = k \} = \Pr \bigcap_{i \in [n]} \{ N_{A_i} = k_i \} \Pr\{N_B = k\} \prod_{i \in [n]} \Pr\{N_{A_i} = k_i\}.
\]

**Proposition 1.2.** For a Poisson process \( \{ N(t) \in [0, \infty) : t \in [n] \} \), distribution of first arrival instant \( S_1 \) conditioned on \( \{ N(t) = 1 \} \) is uniform between \([0, t)\).

**Proof.** If \( N(t) = 1 \), then we know that conditional distribution of \( S_1 \) is supported on \([0, t)\). By Proposition ??, we see that

\[
\Pr\{S_1 \leq s | N(t) = 1\} = \frac{\Pr\{N(s) = 1, N(t - s) = 0 | N(t) = 1\} 1_{s < t}}{\Pr\{N = 1\}} = \frac{s}{t} 1_{s < t}.
\]

**Alternative proof.** For any \( 0 \leq u < t \), we can write \( \{ S_1 = u, N(t) = 1 \} \) as intersection of two independent events,

\[
\{ S_1 = u, N(t) = 1 \} \iff \{ S_1 = u \} \cap \{ X_2 > t - u \}.
\]

Therefore, integrating LHS with respect to \( u \) in interval \([0, s]\) for \( s < t \), we obtain

\[
\Pr\{S_1 \leq s, N(t) = 1\} = \int_0^s du \lambda \exp(-\lambda u) \exp(-\lambda(t - u)) = s \lambda \exp(-\lambda t).
\]

Since \( \Pr\{N(t) = 1\} = \lambda t \exp(-\lambda t) \), it follows that

\[
\Pr\{S_1 \leq s | N(t) = 1\} = \begin{cases} \frac{s}{t}, & s < t \\ 0, & s \geq t. \end{cases}
\]
Proposition 1.3. For a Poisson process \( \{N(t), t \geq 0\} \), joint distribution of arrival instant \( \{S_1, \ldots, S_n\} \) conditioned on \( \{N(t) = n\} \) is identical to joint distribution of order statistics of \( \text{iid} \) uniformly distributed random variables between \([0, t]\).

Proof. Let \( \{s_0 = 0 < s_1 < s_2 < \cdots < s_n < t\} \) be a finite sequence of non-negative increasing numbers between 0 and \( t \). Then, by Proposition 2.1, we get
\[
\Pr \left( \bigcap_{i \in [n]} \{S_i \leq s_i\} | N(t) = n \right) = \Pr \left( \bigcap_{i \in [n]} \{N([0, s_i]) \geq i\} | N(t) = n \right).
\]

Alternative proof. Let \( \{s_i \in (0, t) : i \in [n]\} \) be a sequence of increasing numbers. If we denote \( s_0 = 0 \), then we can write
\[
\bigcap_{i=1}^{n} \{S_i = s_i\} \cap \{N(t) = n\} \iff \bigcap_{i=1}^{n} \{X_i = s_i - s_{i-1}\} \cap \{X_{n+1} > t - s_n\}.
\]

Note that all the events on RHS are independent events. Therefore, it is easy to compute the joint distribution of \( \{S_1, \ldots, S_n\} \), as
\[
\Pr \left( \bigcap_{i=1}^{n} \{S_i \leq s_i\} \cap \{N(t) = n\} \right) = \int_{0}^{s_1} du_1 \cdots \int_{0}^{s_n} du_n \prod_{i=1}^{n} \lambda \exp(-\lambda(u_i - u_{i-1}) \exp(-\lambda(t - u_n))
\]
\[
= \lambda^n \exp(-\lambda t) \prod_{i=1}^{n} s_i.
\]

Since \( \Pr\{N(t) = n\} = \exp(-\lambda t) \lambda^n / n! \), it follows that
\[
\Pr\{S_1 \leq s_1, \ldots, S_n \leq s_n | N(t) = n\} = \begin{cases} n! \prod_{i=1}^{n} \frac{s_i}{t} & s < t \\ 0 & s \geq t. \end{cases}
\]

Let \( U_1, \ldots, U_n \) are iid Uniform random variables in \([0, t]\). Then, the order statistics of \( U_1, \ldots, U_n \) has an identical joint distribution to \( n \) arrival instants conditioned on \( \{N(t) = n\} \). \( \square \)

2 Age and excess time

Definition 2.1. For a point process \( \{N(t), t \geq 0\} \), we can define age process \( \{A(t), t \geq 0\} \) and excess time process \( \{Y(t), t \geq 0\} \) as
\[
A(t) = t - S_{N(t)}, \quad Y(t) = S_{N(t)+1} - t.
\]

Proposition 2.2. For a Poisson process with rate \( \lambda \), the corresponding age and excess time are both exponentially distributed with rate \( \lambda \) irrespective of time \( t \).

Proof. Using stationary independent increment property of Poisson process, we can write complementary distribution of excess time process as
\[
\Pr\{Y(t) > y\} = \sum_{n \in \mathbb{N}_0} \Pr\{Y(t) > y, N(t) = n\} = \sum_{n \in \mathbb{N}_0} \Pr\{N(t + y) - N(t) = 0, N(t) = n\}
\]
\[
= \Pr\{N(y) = 0\} \sum_{n \in \mathbb{N}_0} \Pr\{N(t) = n\} = \Pr\{N(y) = 0\}.
\]

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Similarly, we can write complementary distribution for the age process as
\[
\Pr\{A(t) \geq x\} = \sum_{n \in \mathbb{N}_0} \Pr\{A(t) \geq x, N(t) = n\} = \sum_{n \in \mathbb{N}_0} \Pr\{N(t) - N(t - x) = 0, N(t) = n\} = \sum_{n \in \mathbb{N}_0} \Pr\{N(t - x) = n\} \Pr\{N(x) = 0\} = \Pr\{N(x) = 0\}.
\]

\[
\square
\]

3 Superposition and decomposition of Poisson processes

**Theorem 3.1 (Sum of Independent Poissons).** Let \( \{N_1(t), t \geq 0\} \) and \( \{N_2(t), t \geq 0\} \) be two independent Poisson processes with rates \( \lambda_1 \) and \( \lambda_2 \) respectively. Then, the process \( N(t) = N_1(t) + N_2(t) \) is Poisson with rate \( \lambda_1 + \lambda_2 \).

**Proof.** We need to show that \( \{N(t)\} \) has stationary independent increments, and
\[
\Pr\{N(t) = n\} = \exp(-\lambda_1 - \lambda_2)t)\frac{(\lambda_1 + \lambda_2)^n t^n}{n!}.
\]

For two disjoint interval \((t_1, t_2)\) and \((t_3, t_4)\), we can see that for both processes \(N_1(t)\) and \(N_2(t)\), arrivals in \((t_1, t_2)\) and \((t_3, t_4)\) are independent. Therefore, \(N(t)\) has independent increment property. Similarly, we can argue about the stationary increment property of \(\{N(t)\}\). Further, we can write
\[
\{N(t) = n\} = \bigcup_{k=0}^{n} \{N_1(t) = k\} \cap \{N_2(t) = n - k\}.
\]

Since \(N_1(t)\) and \(N_2(t)\) are independent, we can write
\[
\Pr\{N(t) = n\} = \sum_{k=0}^{n} \exp(-\lambda_1 t)\frac{(\lambda_1 t)^k}{k!} \exp(-\lambda_2 t)\frac{(\lambda_2 t)^{n-k}}{(n-k)!} = \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} (\lambda_1 t)^k (\lambda_2 t)^{n-k}.
\]

Result follows by recognizing that summand is just binomial expansion of \([\lambda_1 + \lambda_2]t^n\).

**Remark 3.2.** If independence condition is removed, the statement is not true.

**Theorem 3.3 (Independent Splitting).** Let \( \{N(t), t \geq 0\} \) be a Poisson arrival process. Each arrival can be randomly assigned to either arrival type 1 or 2, with probability \( p \) and \( 1 - p \) respectively, independent of previous assignments. Arrival processes of type 1 and 2 are denoted by \( N_1(t) \) and \( N_2(t) \) respectively. Then, \( \{N_1(t), t \geq 0\} \) and \( \{N_2(t), t \geq 0\} \) are mutually independent Poisson processes with rates \( \lambda p \) and \( \lambda (1 - p) \) respectively.

**Proof.** To show that \( N_1(t), t \geq 0 \) is a Poisson process with rate \( \lambda p \), we show that it is stationary independent increment process with the distribution
\[
\Pr\{N_1(t) = n\} = \frac{(p\lambda t)^n}{n!} e^{-\lambda pt}.
\]
Figure 1: Splitting a Poisson process into two independent Poisson processes.

The stationary, independent increment property of the probabilistically filtered processes \( \{N_1(t), t \geq 0\} \) and \( \{N_2(t), t \geq 0\} \) can be understood and argued out from the example given in the figure. Notice that

\[
\{N_1(t) = k\} = \bigcup_{n=k}^{\infty} \{N(t) = n, N_1(t) = k\}.
\]

Further notice that conditioned on \( \{N(t) = n\} \), probability of event \( \{N_1(t) = k\} \) is merely probability of selecting \( k \) arrivals out of \( n \), each with independent probability \( p \). Therefore,

\[
\Pr\{N_1(t) = k\} = \exp(-\lambda t) \sum_{n=k}^{\infty} \frac{(\lambda t)^n}{n!} \binom{n}{k} p^k (1-p)^{n-k},
\]

\[
= \exp(-\lambda t) \left(\frac{\lambda p t}{k!}\right) \sum_{n=k}^{\infty} \frac{(\lambda (1-p)t)^{n-k}}{(n-k)!}.
\]

Recognizing that infinite sum in RHS adds up \( \exp(\lambda (1-p)t) \), the result follows. We can find the distribution of \( N_2(t) \) by similar arguments. We will show that events \( \{N_1(t) = n_1\} \) and
\{N_2(t) = n_2\} are independent. To this end, we see that
\[
\{N_1(t) = n_1, N_2(t) = n_2\} = \{N(t) = n_1 + n_2, N_1(t) = n_1\}.
\]
Using their distribution for \(N_1(t), N_2(t)\), and conditional distribution of \(N_1(t)\) on \(N(t)\), we can show that
\[
\Pr\{N_1(t) = n_1, N_2(t) = n_2\} = \exp(-\lambda t) \frac{(\lambda t)^{n_1+n_2}}{(n_1+n_2)!} \left(\frac{n_1+n_2}{n_1}\right) p^{n_1}(1-p)^{n_2},
\]
\[
= \Pr\{N_1(t) = n_1\} \Pr\{N_2(t) = n_2\}.
\]
In general, we need to show finite dimensional distributions factorize. That is, we need to show that for measurable sets \(A_1, \ldots, A_n : j \in [m]\), we have
\[
\Pr\left(\bigcap_{i=1}^n \{N_1(t_i) \in A_i\} \bigcap_{j=1}^m \{N_2(s_j) \in B_j\}\right) = \Pr\left(\bigcap_{i=1}^n \{N_1(t_i) \in A_i\}\right) \Pr\left(\bigcap_{j=1}^m \{N_2(s_j) \in B_j\}\right).
\]
\[\square\]