Homework 2

1. **Problem 1**
   Consider the composite hypothesis testing problem:
   \[ H_0 : Y \text{ has density } p_0(y) = \frac{1}{2} e^{-|y|}, \quad y \in \mathbb{R} \]
   versus
   \[ H_1 : Y \text{ has density } p_1(y) = \frac{1}{2} e^{-|y-\theta|}, \quad y \in \mathbb{R}, \theta > 0. \]
   (a) Describe the locally most powerful \( \alpha \)-level test and derive its power function.
   (b) Does a uniformly most powerful test exist? If so, find it and derive its power function.
   If not, find the generalized likelihood ratio test for \( H_0 \) versus \( H_1 \).

2. **Problem 2**
   Consider the following pair of hypotheses concerning a sequence \( Y_1, Y_2, \ldots, Y_n \) of independent random variables,
   \[ H_0 : Y_k \sim \mathcal{N}(0, \sigma^2), \quad k = 1, 2, \ldots, n \]
   versus
   \[ H_1 : Y_k \sim \mathcal{N}(\mu, \sigma^2), \quad k = 1, 2, \ldots, n \]
   where \( \mu \) is a known constant and \( \sigma > 0 \) is unknown.
   Does there exist a uniformly most powerful test. If so, find it and show that it is UMP. If not, show why and find the generalized likelihood ratio test.

3. **Problem 3**
   Consider the random sample \( X_1, X_2, \ldots, X_n \) with \( X_i \sim \mathcal{N}(\mu, \sigma^2) \), and the statistic \( T_1(X) = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \), where \( \bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i \).
   (a) Consider \( \sigma^2 \) is known, and \( \mu \) is unknown. Derive the expression for the minimum number of samples \( n \) required for \( \mathbb{P}(|T_1(X)| \leq \alpha) \geq \gamma \), where \( \alpha > 0, \) and \( 0 \leq \gamma \leq 1 \).
   (b) Consider \( \sigma^2 \) also to be unknown, and derive the expression for the minimum number of samples \( n \) required for the \( t \)-statistic defined in class \( \mathbb{P}(|T_2(X)| \leq \alpha) \geq \gamma \), where \( \alpha > 0, \) and \( 0 \leq \gamma \leq 1 \).
   (c) Programming exercise: Plot the distribution of the \( T_1 \) and \( T_2 \) statistics for \( n \in \{1000, 10000, 50000\} \), for the case \( X_i \sim \mathcal{N}(2, 2) \). (Hint: Generate several random samples of size \( n \) from the given distribution and plot the histogram).
   (d) Verify the results from (a), (b) above using the distributions computed in (c).
4. **Problem 4**
The measurements from an experiment with \( n \) independent trials follow the below model,

\[
y_k = \alpha x_k + \epsilon_k, \quad k = 1, 2, \ldots, n
\]

where \( \{x_k\} \) are known parameters which depend on the observation instance, and \( \epsilon_k \) is the noise in the measurement process. Assume noise to be independent and identically distributed Gaussian distributed with mean zero and some unknown standard deviation \( \sigma \).

(a) Consider the statistic \( \hat{\alpha} = \frac{1}{\sum_{k=1}^{n} x_k^2} \sum_{k=1}^{n} y_k x_k \). Derive the distribution of \( \hat{\alpha} \).

(b) Programming exercise: Generate random samples from the measurement model described above with \( x_k = k \), and \( \sigma = 1 \), compute the statistic \( \hat{\alpha} \), and find the minimum number of samples \( n \) required for \( |\hat{\alpha} - \alpha| < 0.99 \).

5. **Problem 5**
Consider the M-ary decision problem: \((\Gamma = \mathbb{R}^n)\)

\[
\begin{align*}
H_0 : \ Y & = N + \xi_0 \\
H_1 : \ Y & = N + \xi_1 \\
& \vdots \\
H_{M-1} : \ Y & = N + \xi_M
\end{align*}
\]

where \( \xi_0, \xi_1, \ldots, \xi_M \) are known signals with equal energies, \( \|\xi_0\|^2 = \|\xi_1\|^2 = \cdots = \|\xi_{M-1}\|^2 \).

(a) Assuming \( N = \mathcal{N}(0, \sigma^2 \mathbf{I}) \), find the decision rule achieving minimum error probability when all hypotheses are equally likely.

(b) Assuming further that the signal are orthogonal, show that minimum error probability is given by

\[
P_e = 1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [\Phi(x)]^{M-1} e^{-(x-d)^2/2} \, dx
\]

where \( d^2 = \|\xi_0\|^2/\sigma^2 \), \( x = \frac{\xi_0^T Y}{(\sigma \|\xi_0\|)} \).

6. **Problem 6**
Consider an observed random \( n \)-vector \( \mathbf{Y} \) that satisfies one of the two hypotheses:

\[
\begin{align*}
H_0 : \ Y & = N, \\
versus \\
H_1 : \ Y & = N + A \left[ (1 - \Theta)\xi^{(0)} + \Theta \xi^{(1)} \right],
\end{align*}
\]

where \( N = \mathcal{N}(0, \mathbf{I}) \); the quantity \( A \) is a nonrandom positive scalar; the random parameter \( \Theta \) is independent of \( N \) and takes on the values 0 and 1 with equal probabilities; and the signals \( \xi^{(0)} \) and \( \xi^{(1)} \) are known orthonormal signals.

(a) Suppose the value of \( A \) is known. Find the likelihood ratio between the hypotheses \( H_0 \) and \( H_1 \).
(b) Consider now the composite hypothesis-testing problem:

\[ H_0 : A = 0, \]

versus

\[ H_1 : A > 0. \]

Find the locally most powerful test of level \( \alpha \). Draw the corresponding detector structure.

7. **Problem 7**

Consider the model

\[ Y_k = \theta^{1/2} s_k R_k + N_k, \quad k = 1, 2, \ldots, n \quad (1) \]

where \( s_1, s_2, \ldots, s_n \) is a known signal sequence, \( \theta \geq 0 \) is a constant, and \( R_1, R_2, \ldots, R_n, N_1, N_2, \ldots, N_n \) are i.i.d. \( \mathcal{N}(0, 1) \) random variables

(a) Consider the hypothesis pair,

\[ H_0 : \theta = 0, \]

versus

\[ H_1 : \theta = A \]

where \( A \) is a known positive constant. Describe the structure of the Neyman-Pearson detector.

(b) Consider the hypothesis pair,

\[ H_0 : \theta = 0, \]

versus

\[ H_1 : \theta > 0. \]

Under what conditions on \( s_1, s_2, \ldots, s_n \) does a UMP exist?

(c) For the hypothesis pair of part (b) with \( s_1, s_2, \ldots, s_n \) general, is there a locally optimum detector? If so find it. If not, describe the generalized likelihood ratio test.

8. **Problem 8**

Let \( X_1, \ldots, X_n \) be i.i.d samples from a Poisson(\( \lambda \)) distribution. What is a nontrivial sufficient statistic for \( \lambda \)?