1. (a) Let \((G, +)\) be an abelian group, and \(H\) a subgroup of \(G\). A coset of \(H\) in \(G\) is a subset of the form \(g + H = \{ g + h : h \in H \}\) for some fixed \(g \in G\). Show that the distinct cosets of \(H\) in \(G\) together form a partition of \(G\) into equal-sized subsets. Deduce from this that if \(G\) is a finite group, then \(|H|\) divides \(|G|\), and \(H\) has \(|G|/|H|\) many distinct cosets in \(G\).

(b) In a binary linear code \(C\), show that either all codewords have even Hamming weight, or exactly half of the codewords have even Hamming weight. [Hint: First observe that the codewords of even weight form a subgroup of \(C\).]

2. We would like to construct a binary linear code of blocklength 10 that can, under minimum-distance decoding, correct all single-error patterns, as well as all double-error patterns in which both ones occur within the first five positions (for example, 1100000000, 1000100000 and 0001100000, but not 1000010000 or 0000011000). Other double-error patterns and all higher-order error patterns need not be correctable.

Construct a code with the largest possible dimension that meets the given requirements. You must justify the fact that the dimension of the code you construct is indeed the largest possible under the conditions of the problem. [Hint: How many distinct cosets should a code with the desired error-correcting capability have?]

3. Consider the \([5,2]\) binary linear code with systematic generator matrix

\[
G = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}
\]

A 2-bit message vector \(u = [u_1, u_2]\) is encoded as \(c = uG = [u_1, u_2, p_3, p_4, p_5]\), and transmitted across a memoryless binary symmetric channel with crossover probability \(\epsilon\).

(a) Choose as coset leaders the all-zero word, all words of weight 1, and the words 00011 and 10001. Construct the standard array, together with the syndromes corresponding to each coset.

(b) If the standard array as above is used for decoding, what is the probability that the transmitted codeword \(c\) is decoded incorrectly?

(c) If the code is used only for error detection, what is the probability that any channel errors that take place during transmission go undetected?

(d) If the standard array of part (a) is used for decoding, what is the probability that the first message bit, \(m_1\), is decoded incorrectly?