1. \( C = \{(c_2, c_2, \ldots, c_{10}) \in \mathbb{F}_{11}^{10} : \sum_{i=1}^{10} ic_i \equiv 0 \pmod{11}\} \). From the definition, it is clear that a parity-check matrix for \( C \) is:

\[
H = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{bmatrix}.
\]

Clearly, rank(\( H \)) = 1.

(a) By the rank-nullity theorem, \( \dim(C) = n - \text{rank}(H) = 10 - 1 = 9 \).

(b) Yes, it has a systematic generator matrix. To see this, note that multiplying the \( H \) above by 10 (mod 11) gives another parity-check matrix

\[
H' = \begin{bmatrix} 10 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{bmatrix},
\]

which is in the form \([A \mid I_{n-k}]\). Hence, as proved in class, \( G = [I_k \mid -A^T] \) is a systematic generator matrix for \( C \):

\[
G = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 4 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 5 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 6 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 7 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 8 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 9 \end{bmatrix}.
\]

2. We count the number of ways we can choose row vectors \( g_1, g_2, \ldots, g_k \) such that they form a basis for \( C \).

For \( g_1 \), we can choose any non-zero codeword from \( C \). Thus, the number of ways of choosing \( g_1 \) is \( q^k - 1 \).

The second row vector \( g_2 \) must be chosen from \( C \) in such a way that it does not lie in the span of \( g_1 \). The number of vectors in the span of \( g_1 \) is \( q \). Hence, the number of ways of choosing the second row vector is \( |C| - q = q^k - q \).

Carrying on this fashion, suppose that we have chosen linearly independent vectors \( g_1, g_2, \ldots, g_{j-1} \), for some \( j \leq k \). The next row vector \( g_j \) must lie in \( C \) but must not be in the span of \( g_1, g_2, \ldots, g_{j-1} \). Thus, there are \( q^k - q^{j-1} \) choices for \( g_j \).

Putting it all together, the total number of distinct generator matrices for \( C \) is given by \( \prod_{j=0}^{k-1}(q^k - q^j) \).
3. (a) \(d_{\text{min}}(C) = 1\), as there is a codeword of weight 1, namely, the third row of \(G\).

(b) \(d_{\text{min}}(C^\perp) = 2\). To see this, recall that \(G\) is a parity-check matrix for \(C^\perp\). Since \(G\) has no non-zero columns, there are no words of weight 1 in \(C^\perp\). The last two columns of \(G\) are identical (and hence, linearly dependent), so 000011 is a codeword of weight 2 in \(C^\perp\).

(c) Using elementary row operations (replace row 2 by row 2 + row 3), we obtain a new generator matrix

\[
\begin{bmatrix}
  1 & 0 & 0 & 1 & 1 & 1 \\
  0 & 1 & 0 & 0 & 1 & 1 \\
  0 & 0 & 1 & 0 & 0 & 0
\end{bmatrix}
\]

This generator matrix is in systematic form \([I_k | A]\). Then,

\[
H = [-A^T | I_{n-k}] = \begin{bmatrix}
  1 & 0 & 0 & 1 & 0 & 0 \\
  1 & 1 & 0 & 0 & 1 & 0 \\
  1 & 1 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

is a parity-check matrix for \(C\).

4. (a) The code \(C\) is easily seen to be linear since if \(c_1\) and \(c_2\) are two arrays from the code, then their sum \(c_1 + c_2\) is also in the code, since adding corresponding even-weight rows from the two arrays yields a row of even weight in the sum, and similarly for columns.

(b) \(\dim(C) = 4\): The code \(C\) consists of vectors \([x_1 \; x_2 \; \ldots \; x_9] \in \mathbb{F}_2^9\), which satisfy the following equations (over \(\mathbb{F}_2\)):

\[
\begin{align*}
x_1 + x_2 + x_3 &= 0 \\
x_4 + x_5 + x_6 &= 0 \\
x_7 + x_8 + x_9 &= 0 \\
x_1 + x_4 + x_7 &= 0 \\
x_2 + x_5 + x_8 &= 0 \\
x_3 + x_6 + x_9 &= 0
\end{align*}
\]

Equivalently, \(C\) is the nullspace of the matrix

\[
H = \begin{bmatrix}
  1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
  1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
  0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
  0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1
\end{bmatrix}
\]

Verify (for example, by bringing into rref form) that \(\text{rank}(H) = 5\). Hence, \(\dim(C) = 9 - 5 = 4\).
\[ d_{\text{min}}(C) = 4: \] It is relatively easy to show that the minimum distance of \( C \) is 4. Since the code is linear, it is enough to show that all nonzero codewords have weight at least 4, and that there exists a codeword of weight 4.

Consider any nonzero array from the code. Suppose that the \((i,j)\)th entry of the array is a 1. Since the \(i\)th row must have even weight, there must be another 1 in the row, say, at position \((i,k), k \neq j\). But now, the \(j\)th and \(k\)th columns must each have another 1 so as to maintain even weight. Thus, any nonzero array from the code must have at least four 1’s. Hence the minimum distance of the code is at least 4. Indeed, the minimum distance is exactly 4, since

\[
\begin{array}{ccc}
1 & 1 & 0 \\
1 & 1 & 0 \\
0 & 0 & 0 \\
\end{array}
\]

is a codeword of weight 4.

5. (a) Let \( c' = (c_1, c_2, \ldots, c_{i-1}, c_{i+1}, \ldots, c_n) \in C \) be obtained by puncturing \( c = (c_1, c_2, \ldots, c_n) \in C \) at the \(i\)th coordinate. Let \( g_1, \ldots, g_k \) and \( g'_1, \ldots, g'_k \) denote the rows of \( G \) and \( G' \), respectively. Note now that

\[ c = \sum_{i=1}^{k} \alpha_i g_i \] (for some \( \alpha_i \)'s in \( \mathbb{F} \)) if and only if \( c' = \sum_{i=1}^{k} \alpha_i g'_i \). Consequently, \( C_i = \text{rowspace}(G') \).

(b) Let \( h_1, h_2, \ldots, h_n \) be the columns of \( H \). Now, \( c^{(i)} = (c_1, c_2, \ldots, c_{i-1}, c_{i+1}, \ldots, c_n) \) is in \( C^{(i)} \) iff \( (c_1, c_2, \ldots, c_{i-1}, 0, c_{i+1}, \ldots, c_n) \) is in \( C \), which in turn happens iff \( c_1 h_1 + c_2 h_2 + \cdots + c_{i-1} h_i + c_{i+1} h_{i+1} + \cdots + c_n h_n = 0 \). Hence, \( H' = [h_1 \ h_2 \ \cdots \ h_{i-1} \ h_{i+1} \ \cdots \ h_n] \) is a parity-check matrix for \( C^{(i)} \).

(c) Puncturing: The punctured code \( C_i \) is linear since it is the rowspace of \( G' \). \( C_i \) clearly has blocklength \( n - 1 \). Now, one column of \( G \) is removed for obtaining \( G' \), so the rank can decrease by at most 1. Hence, \( k_i = \text{rank}(G') \geq k - 1 \).

Finally, deleting the \(i\)th coordinate of a codeword can reduce its Hamming weight by at most 1. Thus, \( d_{\text{min}} \) can decrease by at most 1, i.e., \( d_i \geq d - 1 \).

Shortening: The shortened code \( C^{(i)} \) is linear since it is the nullspace of \( H' \). Again, the blocklength is clearly \( n - 1 \). Since \( H' \) is obtained from \( H \) by deleting one column, its rank cannot be more than the rank of \( H \), i.e., \( \text{rank}(H') \leq \text{rank}(H) = n - k \). Therefore, by the rank-nullity theorem, \( \dim(C^{(i)}) = (n - 1) - \text{rank}(H') \geq (n - 1) - (n - k) = k - 1 \).

To understand how \( d_{\text{min}} \) behaves with shortening, consider the subset \( C' \) of \( C \) consisting of all codewords having a 0 in the \(i\)th coordinate. It is easily verified that \( C' \) is a subcode of \( C \). Hence, \( d_{\text{min}}(C') \geq d_{\text{min}}(C) \), as the minimum-weight codewords of \( C \) may not be in \( C' \). However, \( d_{\text{min}}(C^{(i)}) = d_{\text{min}}(C') \), since \( C^{(i)} \) is obtained from \( C' \) by deleting the \(i\)th coordinate, which is always 0, so that the weight of the codewords is not affected. We conclude that \( d_{\text{min}}(C^{(i)}) \geq d_{\text{min}}(C) \).