This assignment consists of two pages.

1. Consider the ISBN code

\[ C = \{ (c_1, c_2, \ldots, c_{10}) \in \mathbb{F}_{11}^{10} : \sum_{i=1}^{10} ic_i \equiv 0 \pmod{11} \}, \]

which is a linear code over \( \mathbb{F}_{11} \).

(a) Determine the dimension and minimum distance of \( C \).

(b) Does \( C \) have a systematic generator matrix? Justify your answer.

2. Show that a linear code \( C \) of dimension \( k \) over \( \mathbb{F}_q \) has exactly

\[ \prod_{j=0}^{k-1} (q^k - q^j) \]

distinct generator matrices.

[Hint: How many different ways are there of choosing the first row vector of \( G \)? Having chosen the first row vector, in how many ways can you choose the second row, and so on.]

3. Let \( C \) be the binary linear code with generator matrix

\[
G = \begin{bmatrix}
1 & 0 & 0 & 1 & 1 & 1 \\
0 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0
\end{bmatrix}
\]

(a) What is the minimum distance of \( C \)? Justify your answer.

(b) What is the minimum distance of \( C^\perp \)? Justify your answer.

(c) Give a parity-check matrix for \( C \).

4. Let \( C \) be a binary code of length 9 that consists of all \( 3 \times 3 \) arrays of the form

\[
\begin{array}{ccc}
x_1 & x_2 & x_3 \\
x_4 & x_5 & x_6 \\
x_7 & x_8 & x_9 \\
\end{array}
\]

such that each row and column has even Hamming weight.

(a) Verify that \( C \) is a linear code.

(b) Determine the dimension and minimum distance of \( C \).

The above is an example of a product code. In general, the product of a length-\( n_1 \) block code, \( C_1 \), and a length-\( n_2 \) block code, \( C_2 \), is a code, \( C_{pr} \), whose codewords consist of all \( n_1 \times n_2 \) arrays in which the rows belong to \( C_1 \) and the columns to \( C_2 \). It is easy to see that if \( C_1 \) and \( C_2 \) are linear, then so is \( C_{pr} \).
5. Puncturing and shortening. Let $C$ be an $[n, k, d]$ linear code over $F$ with $1 < k < n$. For $i \in \{1, 2, \ldots, n\}$, define $C_i$ to be the code

$$C_i \triangleq \{(c_1, c_2, \ldots, c_{i-1}, c_{i+1}, \ldots, c_n) : (c_1, c_2, \ldots, c_n) \in C\},$$

and define $C^{(i)}$ to be the code

$$C^{(i)} \triangleq \{(c_1, c_2, \ldots, c_{i-1}, c_{i+1}, \ldots, c_n) : (c_1, c_2, \ldots, c_{i-1}, 0, c_{i+1}, \ldots, c_n) \in C\}.$$

$C_i$ and $C^{(i)}$ are said to be the codes obtained by, respectively, puncturing and shortening $C$ at the $i$th coordinate.

(a) Show that if $G'$ is obtained by deleting the $i$th column from a generator matrix $G$ of $C$, then $C_i = \text{rowspace}(G')$.

(b) Show that if $H'$ is obtained by deleting the $i$th column from a parity-check matrix $H$ of $C$, then $C^{(i)} = \text{nullspace}(H')$. (In other words, a matrix obtained by deleting the $i$th column from any parity-check matrix of $C$ is a parity-check matrix for $C^{(i)}$.)

(c) Argue that $C_i$ is an $[n - 1, k_i, d_i]$ linear code with $k_i \geq k - 1$ and $d_i \geq d - 1$, and that $C^{(i)}$ is an $[n - 1, k^{(i)}, d^{(i)}]$ linear code with $k^{(i)} \geq k - 1$ and $d^{(i)} \geq d$. 