1. Let \( \{N(t)\}_{t \geq 0} \) be a Poisson process of rate \( \beta > 0 \), and let \( Z_k \) denote the \( k \)th arrival time of the process. Prove that \( \lim_{k \to \infty} Z_k = \infty \) with probability 1.

*Hint:* For positive integers \( k \) and \( N \), define the event \( A_{k,N} = \{Z_k > N\} \), and let \( A_N \) denote the event \( \{A_{k,N} \text{ occurs for all sufficiently large } k\} \). In other words, \( A_N = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_{k,N} \). Use the Borel-Cantelli lemma to argue that \( P(\bigcap_{N=1}^{\infty} A_N) = 0 \). Now, complete the proof using the fact that the event \( \{\lim_{k \to \infty} Z_k = \infty\} \) can be expressed as \( \bigcap_{N=1}^{\infty} A_N \).

2. Packets arrive at the input of a channel according to a Poisson process of rate \( \lambda \), with \( 0 < \lambda < \infty \). If the channel is empty when a packet arrives, it accepts the packet. The packet then stays in the channel for an amount of time \( T \) that is exponentially distributed with parameter \( \mu \), with \( 0 < \mu < \infty \), i.e.,

\[
P(T > t) = e^{-\mu t} \quad \text{for all } t > 0.
\]

The time \( T \) is independent of the packet arrivals process. The channel can only hold one packet at a time, and there is no buffer at the channel input, so if a new packet arrives while the channel is occupied, the new packet gets dropped. Let \( X(t) \in \{0, 1\} \) denote the number of packets in the channel at time \( t \). It is easily seen that \( \{X(t)\}_{t \geq 0} \) is a continuous-time Markov chain (CTMC) on the state space \( \{0, 1\} \) (you do not need to verify this).

(a) Determine the sojourn time parameters \( a_0, a_1 \), the transition probability matrix \( P \) of the embedded DTMC, and the transition rate matrix \( Q \) of the CTMC.

(b) Is the CTMC regular? You must justify your answer.

(c) For \( j \in \{0, 1\} \), let \( S_{jj} \) denote the recurrence time, i.e., starting in state \( j \), the time it takes for the CTMC to leave state \( j \) and then return to it for the first time. Determine \( E[S_{jj}] \) for \( j \in \{0, 1\} \).

(d) Does the CTMC have a stationary distribution? If so, what is it?

3. Consider a birth-death process with \( \lambda_j = \mu_j = \gamma^{-j} \) for all \( j \geq 1 \), where \( \gamma \in (0, 1) \) is a constant. (Take \( \lambda_0 = 1 \).) Prove that the birth-death process is positive recurrent, while its embedded DTMC is null recurrent.

4. Consider a birth-death process with \( \lambda_j = \gamma^j \) for all \( j \geq 0 \), and \( \mu_j = \gamma^{j-1} \) for all \( j \geq 1 \), where \( \gamma \in (0, 1) \) is a constant. Prove that the birth-death process is null recurrent, while its embedded DTMC is positive recurrent.

*Hint:* To show that the embedded DTMC is positive recurrent, solve the reversibility conditions to obtain a stationary distribution.