1. Basic properties of variance.
   (a) Verify that \( \text{Var}(aX) = a^2 \text{Var}(X) \) for any \( a \in \mathbb{R} \).
   (b) Show that \( \text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y) + 2 \text{Cov}(X, Y) \).

2. Moment generating functions.
   (a) Determine the moment generating function of a random variable \( X \) having a \( \text{GEOM}(p) \) distribution, i.e., \( \mathbb{P}(X = k) = (1 - p)^{k-1} p \) for \( k = 1, 2, 3, \ldots \).

   Use the MGF to compute the first two moments of \( X \).

   (b) Verify that for an r.v. \( X \) taking values in \( \{ \pm 1, \pm 2, \pm 3, \ldots \} \), with \( \mathbb{P}(X = k) = (3/\pi)^{1/k^2} \), for \( k = \pm 1, \pm 2, \pm 3, \ldots \), the moment generating function \( M_X(t) \) is not finite for \( t \neq 0 \).

3. Characteristic functions.
   - Let \( \Phi_X \) denote the characteristic function of a random variable \( X \). Let \( W = a + X \) and \( Y = aX \) for some constant \( a \in \mathbb{R} \). Express the characteristic functions \( \Phi_W \) and \( \Phi_Y \) in terms of \( \Phi_X \).
   - Recall that the pdf of a Gaussian random variable \( Z \) with mean 0 and variance 1 is given by \( f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}, \ z \in \mathbb{R} \).

   Compute the characteristic function of \( Z \).
   - From (a) and (b), deduce the characteristic function of a Gaussian rv with mean \( \mu \) and variance \( \sigma^2 \).

4. Let \( X_1, X_2, \ldots, X_n \) be \( n \) independent random variables, with each \( X_i \) satisfying \( \mathbb{P}(X_i = +1) = \mathbb{P}(X_i = -1) = 1/2 \).

   (a) Determine the mean and variance of \( \bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i \).
   (b) Use Chebyshev’s inequality to bound the probability \( \mathbb{P}(|\bar{X}| > \delta) \) for \( \delta > 0 \).
   (c) Use the Chernoff bounding technique to show that for \( 0 < \delta < 1 \),

   \[
   \mathbb{P}(\bar{X} > \delta) \leq 2^{-n[1-h(\frac{\delta}{2\delta})]},
   \]

   where \( h(\cdot) \) is the binary entropy function defined by \( h(x) = -x \log_2(x) - (1 - x) \log_2(1 - x) \) for \( x \in [0, 1] \). Hence, by symmetry,

   \[
   \mathbb{P}(|\bar{X}| > \delta) = 2 \cdot \mathbb{P}(X > \delta) \leq 2 \cdot 2^{-n[1-h(\frac{\delta}{2\delta})]}.
   \]

   The point of this exercise is that the Chernoff bound yields an exponential decay in \( n \) for the tail probability of \( \bar{X} \), while Chebyshev’s inequality only yields a \( 1/n \) decay.