1. Let $\lambda, \gamma > 0$ be given. Suppose that $X_n, n = 1, 2, 3, \ldots$, is a sequence of i.i.d. exponential random variables with mean $1/\lambda$, and that $N$ is a geometric random variable independent of $(X_n)_{n \geq 1}$, with mean $1/\gamma$. Prove that $X = \sum_{i=1}^{N} X_i$ is an exponential random variable with mean $1/(\gamma \lambda)$. [Hint: Use moment generating functions.]

2. Let $\{N(t)\}_{t \geq 0}$ be a Poisson process of rate $\beta > 0$, and let $Z_k$ denote the $k$th arrival time of the process. Prove that $\lim_{k \to \infty} Z_k = \infty$ with probability 1. [Hint: For positive integers $k$ and $N$, define the event $A_{k,N} = \{Z_k > N\}$, and let $A_N$ denote the event $\{A_{k,N} \text{ occurs for all sufficiently large } k\}$. In other words, $A_N = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_{k,N}$. Use the Borel-Cantelli lemma to argue that $P\left((A_N)^c\right) = 0$. Now, complete the proof using the fact that the event $\{\lim_{k \to \infty} Z_k = \infty\}$ can be expressed as $\bigcap_{N=1}^{\infty} A_N$.]

3. Let $\{N_1(t), t \geq 0\}$ and $\{N_2(t), t \geq 0\}$ be two independent Poisson processes with strictly positive rates $\lambda_1$ and $\lambda_2$, respectively. To ensure their time-homogeneity as continuous-time Markov chains, we do not require that $N_1(0) = 0$ and $N_2(0) = 0$. Set $X(t) = N_1(t) - N_2(t)$, for all $t \geq 0$.

(a) Show that $\{X(t)\}$ has the independent increments property, and hence, is a continuous-time Markov chain. What is its state space, $S$?

(b) Determine the transition probability matrix $P$ of the embedded discrete-time Markov chain, and the parameters $a_i$ of the sojourn times associated with the states $i \in S$.

(c) Is $\{X(t)\}$ regular?

(d) Is $\{X(t)\}$ irreducible? recurrent?