Index Coding using Interference Management

Niranjana Ambadi

Department of Electrical Communication Engineering
Indian Institute of Science, Bangalore - India

ECE Students Seminar Series, October 2017.
Index Coding Problem
Single sender, Multiple users

- Single source transmitting messages from a finite alphabet \( \mathcal{M} = \{x_1, x_2, \ldots, x_M\} \) to \( K \) receivers/destinations
- A receiver \( D_i \triangleq (W_i, A_i) \) wants \( W_i \subseteq \mathcal{M} \) and knows \( A_i \subseteq \mathcal{M} \) a priori as side-information
- **Noiseless Index Coding Problem**: Identify the minimum number of transmissions (optimal length) so that all the receivers can decode their wanted messages using the transmitted symbols and their prior information.\(^1\)

---
Topological Interference Management

Figure: TIM setting - example
Problem Formulation

Single Unicast Neighboring Interference Symmetric Index Coding

Figure: Interferers and antidotes at destination $D_1$
Sample Problem

Index Coding Problem with $K = 7$ destinations. $D_i \triangleq (\mathcal{W}_i, \mathcal{A}_i)$.

<table>
<thead>
<tr>
<th>Demand Set, $\mathcal{W}_i$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
<th>$x_6$</th>
<th>$x_7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Side-information, $\mathcal{A}_i$</td>
<td>$x_4, x_5$</td>
<td>$x_5, x_6$</td>
<td>$x_6, x_7$</td>
<td>$x_7, x_1$</td>
<td>$x_1, x_2$</td>
<td>$x_2, x_3$</td>
<td>$x_3, x_4$</td>
</tr>
</tbody>
</table>

- Optimal length (minrank): ???
Sample Problem

Index Coding Problem with $K = 7$ destinations. $D_i \triangleq (W_i, A_i)$.

<table>
<thead>
<tr>
<th>Demand Set, $W_i$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
<th>$x_6$</th>
<th>$x_7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Side-information, $A_i$</td>
<td>$x_4, x_5$</td>
<td>$x_5, x_6$</td>
<td>$x_6, x_7$</td>
<td>$x_7, x_1$</td>
<td>$x_1, x_2$</td>
<td>$x_2, x_3$</td>
<td>$x_3, x_4$</td>
</tr>
</tbody>
</table>

- Optimal length (minrank): ???
- Index code: ???
**Definitions**

### Interferers

*Interferers:* For each destination $D_i \in \mathcal{D}$ the set of interfering messages is given by $\mathcal{I}_i = (\mathcal{W}_i \cup \mathcal{A}_i)^c$.

<table>
<thead>
<tr>
<th>Demand Set, $\mathcal{W}_i$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
<th>$x_6$</th>
<th>$x_7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Side-information, $\mathcal{A}_i$</td>
<td>$x_4$, $x_5$</td>
<td>$x_5$, $x_6$</td>
<td>$x_6$, $x_7$</td>
<td>$x_7$, $x_1$</td>
<td>$x_1$, $x_2$</td>
<td>$x_2$, $x_3$</td>
<td>$x_3$, $x_4$</td>
</tr>
<tr>
<td>Interferers, $\mathcal{I}_i$</td>
<td>$x_2$, $x_3$</td>
<td>$x_3$, $x_4$</td>
<td>$x_4$, $x_5$</td>
<td>$x_5$, $x_6$</td>
<td>$x_6$, $x_7$</td>
<td>$x_7$, $x_1$</td>
<td>$x_1$, $x_2$</td>
</tr>
</tbody>
</table>
Definitions

Scalar Linear Codes

**Scalar Linear Index Code:** When $S$ is a finite field, an $(S, n, R)$ index code is scalar linear if, for the source with $M$ messages, $\mathcal{M} = \{x_1, x_2, \ldots, x_M\}$, the transmitted symbol sequence is given by,

$$S^n = \sum_{j=1}^{M} V_j x_j.$$ 

The $n \times 1$ vector $V_j$ - the precoding vector (or beamforming vector) for the message $x_j$.

\[
\begin{bmatrix}
0 \\
1 \\
0 \\
0
\end{bmatrix} x_1 +
\begin{bmatrix}
0 \\
0 \\
1 \\
0
\end{bmatrix} x_2 +
\begin{bmatrix}
0 \\
0 \\
0 \\
1
\end{bmatrix} x_3 +
\begin{bmatrix}
1 \\
0 \\
0 \\
0
\end{bmatrix} x_4 +
\begin{bmatrix}
0 \\
1 \\
0 \\
0
\end{bmatrix} x_5 +
\begin{bmatrix}
0 \\
0 \\
1 \\
0
\end{bmatrix} x_6 +
\begin{bmatrix}
1 \\
0 \\
0 \\
0
\end{bmatrix} x_7
\]
Definitions

Symmetric Index Coding

**Symmetric Index Coding**: An index coding problem is symmetric if for any two receivers $D_i$ and $D_j$, $i, j \in \left[ K \right]; j \neq i$ there exists

1. a bijection $\pi : A_i \to A_j$ such that $\pi(x_k) = x_{k+j-i}$; and
2. a bijection $\omega : W_i \to W_j$ such that $\omega(x_k) = x_{k+j-i}$.

**In simple Terms!!!** Relative to its index, each destination has identical sets of wanted messages and side-information.
Sample Problem

Index Coding Problem with $K = 7$ destinations. $D_i \triangleq (\mathcal{W}_i, \mathcal{A}_i)$.

<table>
<thead>
<tr>
<th>Demand Set, $\mathcal{W}_i$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
<th>$x_6$</th>
<th>$x_7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Side-information, $\mathcal{A}_i$</td>
<td>$x_4, x_5$</td>
<td>$x_5, x_6$</td>
<td>$x_6, x_7$</td>
<td>$x_7, x_1$</td>
<td>$x_1, x_2$</td>
<td>$x_2, x_3$</td>
<td>$x_3, x_4$</td>
</tr>
</tbody>
</table>
Sample Problem

Index Coding Problem with $K = 7$ destinations. $D_i \triangleq (\mathcal{W}_i, \mathcal{A}_i)$.

<table>
<thead>
<tr>
<th>Demand Set, $\mathcal{W}_i$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
<th>$x_6$</th>
<th>$x_7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Side-information, $\mathcal{A}_i$</td>
<td>$x_4$, $x_5$</td>
<td>$x_5$, $x_6$</td>
<td>$x_6$, $x_7$</td>
<td>$x_7$, $x_1$</td>
<td>$x_1$, $x_2$</td>
<td>$x_2$, $x_3$</td>
<td>$x_3$, $x_4$</td>
</tr>
</tbody>
</table>

- Optimal length (minrank) = 4,
- Index code: $x_7 + x_4, x_1 + x_5, x_2 + x_6, x_3$. 
Algorithm

- Use interference alignment.
- The messages in a given demand set $\mathcal{W}_i$ must be sent independently.
- The messages interfering at a given $D_i$ that come as part of the demand set of another receiver must be sent independently to each other as well as to the messages in the demand set.

The strategy is...:
Our Contributions

Algorithm

- Use interference alignment.
- The messages in a given demand set $\mathcal{W}_i$ must be sent independently.
- The messages interfering at a given $D_i$ that come as part of the demand set of another receiver must be sent independently to each other as well as to the messages in the demand set.

The strategy is...

"...Count Dimensions needed to avoid Interference."
Finding the optimal length, $\lambda$

Demonstrating the algorithm through example

1. **Finding the interference pattern at each destination:** Pick up any receiver, say $D_1$. Find which receiver shares the maximum number of interferers with $D_1$. Answer: $D_2$.

Next find which receiver shares the maximum interferers with both $D_1$ and $D_2$. Answer: $D_3$

Whenever $D \geq U$, we can conclude that the receiver $D_{j+1}$ will have the maximum number of interferers in common with $D_1, D_2, \ldots, D_{j-1}$ for $j < U$.

<table>
<thead>
<tr>
<th>$\mathcal{W}_1$</th>
<th>$\mathcal{A}_1$</th>
<th>$\mathcal{I}_1$</th>
<th>${\mathcal{I}_1 \cap \mathcal{I}<em>2 \cap \mathcal{I}</em>{U+1}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>${x_1}$</td>
<td>${x_4, x_5}$</td>
<td>${x_2, x_3, x_6, x_7}$</td>
<td>$\mathcal{I}_1 \cap \mathcal{I}_2 \cap \mathcal{I}_3 = \phi$</td>
</tr>
</tbody>
</table>
Our Contributions

Algorithm to find the Optimal Length, $\lambda$

Finding the optimal length, $\lambda$

In general, when $D > U$, take
$$I_1 \cap I_2 \cap \cdots \cap I_U \cap I_{a_1} \cap \cdots \cap I_{a_{i-1}} \cap I_j,$$
where
$$a_i \triangleq \arg\{\max_j |I_1 \cap I_2 \cap \cdots \cap I_{a_{i-1}} \cap I_j|; x_j \in I_1 \cap I_2 \cap \cdots \cap I_{a_{i-1}}\}.$$

As there are only a finite number of receivers, the intersection
$$I_1 \cap I_2 \cap I_{a_1} \cap \cdots \cap I_{a_n} \to \emptyset$$
for some destination, say, $D_{an}$.

We compute $z$ as follows:

$$z = |W_1| + |I_1 \cap W_2| + |I_1 \cap I_2 \cap W_3| + \cdots + |I_1 \cap \cdots \cap W_U| + |I_1 \cap \cdots \cap I_U \cap W_{a_1}| + |I_1 \cap \cdots \cap I_U \cap I_{a_1} \cap \cdots \cap I_{a_{n-1}} \cap W_{a_n}|.$$

For the example, $z = |W_1| + |I_1 \cap W_1| + |I_1 \cap I_2 \cap W_3| = 3$
We define $S'_1$ as the set of messages that were taken into account at $D_1$ while computing $z$ as

\[ S'_1 = \{x_1, x_2, \ldots, x_U, x_{a_1}, x_{a_2}, \ldots, x_{a_n}\}. \]  

(1)

Due to the symmetry of the problem, interference pattern is the same at all the receivers. Hence, at $D_i$,

\[ S'_i = \{x_i, x_{2+i-1}, \ldots, x_{a_1+i-1}, \ldots, x_{a_n+i-1}\}. \]  

(2)
Finding the optimal length, $\lambda$

At each destination a set of $z = |S'_1|$ messages with consecutive indices must have linearly independent pre-coding vectors. This is possible only if $K = nz$. If $K \neq nz$, $\lambda$ is defined as $z + 1$, so that the pre-coding vectors of all consecutive sets of $z$ messages can be chosen to be linearly independent. Thus we have,

$$\lambda \triangleq \begin{cases} 
  z + 1, & \text{if } K \neq nz. \\
  z, & \text{if } K = nz. 
\end{cases} \quad (3)$$

For the example: $z = 3, K = 7 \neq nz, \lambda = z + 1 = 4$
Constructing Optimal Scalar Linear Index Code
for Single Unicast Neighboring Interference Symmetric Index Coding

1. The optimal length $\lambda$ is found.

2. Let $\mathcal{T} = \{ T_1, T_2, \ldots, T_\lambda \}$ be the set of columns of the identity matrix $I_{\lambda \times \lambda}$ over $\mathbb{F}_2$. Every message in $S_1 = \{ x_{K-\lambda+D+2}, \ldots, x_K, x_1, x_2, \ldots, x_{D+1} \}$ is assigned a distinct vector in $\mathcal{T}$ as its pre-coding vector. $V_{K-\lambda+D+2} = T_1$, $V_{K-\lambda+D+3} = T_2$, $\ldots$, $V_{D+1} = T_\lambda$.

3. Let $K = q(K - U - 1) + r$.
For $i \in \{ D + 2, D + 3, \ldots, D + r + 1 \}$,

\[
V_i \triangleq V_{i+(K-\lambda)} + V_{i+(K-\lambda)+(K-U-1)} + \cdots + V_{i+K-\lambda+(q-1)(K-U-1)}.
\] (4)
For $i \in \{D + r + 2, D + r + 3, \ldots, K\}$,

$$V_i \triangleq V_{i-(K-\lambda)} + V_{i+(K-\lambda)} + V_{i+(K-\lambda)+(K-U-1)} + \cdots + V_{i+(K-\lambda)+(q-2)(K-U-1)}.$$  \hspace{1cm} (5)

The set of $\lambda$ transmitted symbols of the $(\mathbb{F}_2, \lambda, \mathcal{R})$ index code with $\mathcal{R} = (\frac{1}{\lambda}, \frac{1}{\lambda}, \ldots, \frac{1}{\lambda})$ is given by

$$S^\lambda = \sum_{i=1}^{K} V_i x_i.$$
Sample Problem
Finding $\lambda$ and the optimal code

Example

<table>
<thead>
<tr>
<th>$\mathcal{W}_1$</th>
<th>$\mathcal{A}_1$</th>
<th>$\mathcal{I}_1$</th>
<th>${\mathcal{I}_1 \cap \mathcal{I}<em>2 \cap \mathcal{I}</em>{U+1}}$</th>
<th>$\lambda_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>${x_1}$</td>
<td>${x_4, x_5}$</td>
<td>${x_2, x_3, x_6, x_7}$</td>
<td>$\mathcal{I}_1 \cap \mathcal{I}_2 \cap \mathcal{I}_3 = \phi$</td>
<td>$z =</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\lambda = z + 1 = 4$</td>
<td></td>
</tr>
</tbody>
</table>
## Sample Problem

Finding $\lambda$ and the optimal code

### Example

<table>
<thead>
<tr>
<th>$W_1$</th>
<th>$A_1$</th>
<th>$I_1$</th>
<th>${I_1 \cap I_2 \cap I_{U+1}}$</th>
<th>$\lambda_1$</th>
</tr>
</thead>
</table>
| $\{x_1\}$ | $\{x_4, x_5\}$ | $\{x_2, x_3, x_6, x_7\}$ | $I_1 \cap I_2 \cap I_3 = \phi$ | $\begin{align*} z &= |W_1| + |I_1 \cap W_2| + |I_1 \cap I_2 \cap W_3| \\
&= 1 + 1 + 1 = 3 \\
\lambda &= z + 1 = 4 \end{align*}$ |

- Let $T_1, T_2, T_3, T_4$ be the columns of the $4 \times 4$ identity matrix. Choose $V_7 = T_1, V_1 = T_2, V_2 = T_3, V_3 = T_4$. 
### Sample Problem

**Finding \( \lambda \) and the optimal code**

#### Example

<table>
<thead>
<tr>
<th>( \mathcal{W}_1 )</th>
<th>( \mathcal{A}_1 )</th>
<th>( \mathcal{I}_1 )</th>
<th>( { \mathcal{I}_1 \cap \mathcal{I}<em>2 \cap \mathcal{I}</em>{U+1} } )</th>
<th>( \lambda_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>{( x_1 )}</td>
<td>{( x_4, x_5 )}</td>
<td>{( x_2, x_3, x_6, x_7 )}</td>
<td>( \mathcal{I}_1 \cap \mathcal{I}_2 \cap \mathcal{I}_3 = \phi )</td>
<td>( z =</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>( = 1 + 1 + 1 = 3 )</td>
<td>( \lambda = z + 1 = 4 )</td>
</tr>
</tbody>
</table>

- Let \( T_1, T_2, T_3, T_4 \) be the columns of the \( 4 \times 4 \) identity matrix. Choose \( V_7 = T_1, V_1 = T_2, V_2 = T_3, V_3 = T_4 \).

\[
S^\lambda = Lx = \begin{bmatrix}
1 & 0 & 0 & 0 & - & - & - \\
0 & 1 & 0 & 0 & - & - & - \\
0 & 0 & 1 & 0 & - & - & - \\
0 & 0 & 0 & 1 & - & - & - \\
\end{bmatrix}
\begin{bmatrix}
x_7 \\
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
x_6 \\
\end{bmatrix}
\]
Now $K = q(K - U - 1) + r$. Here $7 = 1(7 - 2 - 1) + 3$. Using (4), $V_4 = V_4 + (7 - 4) = V_7$, $V_5 = V_5 + (7 - 4) = V_1$, $V_6 = V_6 + (7 - 4) = V_2$.

$$S^\lambda = Lx = \begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
x_7 \\
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
x_6
\end{bmatrix}$$
Now $K = q(K - U - 1) + r$. Here $7 = 1(7 - 2 - 1) + 3$. Using (4), $V_4 = V_{4+(7-4)} = V_7$, $V_5 = V_{5+(7-4)} = V_1$, $V_6 = V_{6+(7-4)} = V_2$.

\[ S^\Lambda = Lx = \begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
x_7 \\
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
x_6
\end{bmatrix} \]

The corresponding index code given by

\[ x_7 + x_4, \quad x_1 + x_5, \quad x_2 + x_6, \quad x_3. \]
Our Contributions

Examples

Optimal lengths of consecutive non-neighboring antidotes single unicast symmetric index coding problem

<table>
<thead>
<tr>
<th>$K = 11$</th>
<th>$K = 11$</th>
<th>$K = 10$</th>
<th>$K = 10$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$U, D = 0, 8$</td>
<td>$U, D = 0, 7$</td>
<td>$U, D = 0, 7$</td>
<td>$U, D = 0, 6$</td>
</tr>
<tr>
<td>$U, D = 1, 7$</td>
<td>$U, D = 1, 6$</td>
<td>$U, D = 1, 6$</td>
<td>$U, D = 1, 5$</td>
</tr>
<tr>
<td>$U, D = 2, 6$</td>
<td>$U, D = 2, 5$</td>
<td>$U, D = 2, 5$</td>
<td>$U, D = 2, 4$</td>
</tr>
<tr>
<td>$U, D = 3, 5$</td>
<td>$U, D = 3, 4$</td>
<td>$U, D = 3, 4$</td>
<td>$U, D = 3, 3$</td>
</tr>
<tr>
<td>$U, D = 4, 4$</td>
<td>$U, D = 3, 4$</td>
<td>$U, D = 3, 4$</td>
<td>$U, D = 3, 3$</td>
</tr>
</tbody>
</table>

$\lambda = 9$

$\lambda = 8$

$\lambda = 8$

$\lambda = 8$

$\lambda = 9$

$\lambda = 9$

$\lambda = 9$

$\lambda = 9$

$\lambda = 9$

$\lambda = 6$

$\lambda = 6$

$\lambda = 6$

$\lambda = 6$

$\lambda = 6$

$\lambda = 6$

$\lambda = 6$

$\lambda = 6$

$\lambda = 6$

$\lambda = 6$

$\lambda = 6$
Neighboring antidotes problem

- The consecutive neighboring antidotes problem setting is as follows: For \( i \in \lfloor K \rfloor \), at receiver \( D_i \), \( \mathcal{W}_i = \{x_i\}, \mathcal{A}_i = \{x_{i+1}, x_{i+2}, \ldots, x_{i+d}\} \); \( d < K \). It is already known \(^2\) that the capacity of this problem is

\[
C = (K - d)^{-1}.
\]

- An optimal scalar linear code can be constructed by considering \( U = 0 \) in the neighboring interference problem. With \( d \) antidotes, the number of interferers is given by \( D = K - d - 1 \). Code construction follows.

Interference alignment perspective allows us to solve many complicated index coding problems easily.

Scalar linear capacity in IC scenarios can be obtained by counting the number of dimensions required to avoid interference.

Algorithm to find capacity in any symmetric index coding problem.

Optimal code construction for Neighboring Interference setting with finite number of users/messages.
Thank You!

Email: ambadi@iisc.ac.in